Essential Spectrum of Internal Waves in the Ocean and Atmosphere: the Relation between the Frequency of the Induced Vibrations and Non-Uniqueness of the Limit Amplitude

Andrei Giniatoulline*

Department of Mathematics, Los Andes University, Colombia, South America

Submission: November 12, 2017; Published: February 19, 2018

*Corresponding author: Andrei Giniatoulline, Associate Professor, Department of Mathematics, Los Andes University, Bogota D.C., Colombia, South America, Tel: 5713394949, Fax: 5713324340, Email: aginiato@uniandes.edu.co

Abstract

For various models of three-dimensional fluid which describe the flows in the Atmosphere and the Ocean, we find a relation between the essential spectrum of normal vibrations of internal waves and non-uniqueness of the limit amplitude of vibrations induced by external mass forces.


Introduction

Let us consider a system of partial differential equations in

\[ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \]

of the following type:

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + A \mathbf{v} + \nabla p &= \mathbf{F}(x,t) \\
\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} &= 0
\end{align*}
\]

Here, \( \mathbf{v} = (v_1, v_2, v_3, v_4, v_5) \) is a velocity field of the fluid, \( p(x,t) \) is the scalar field of the dynamic pressure, \( \mathbf{F}(x,t) \) represents mass forces acting on the fluid, and \( A \) is a matrix operator which defines the specific model.

For example, in the model of rotational motion of the Earth over the vertical axis, the operator \( A \) takes the form

\[
A = \begin{pmatrix}
-\omega_2 & 0 \\
\omega_1 & 0 \\
0 & 0
\end{pmatrix}, \quad n = 3
\]

For the model of stratified fluid with exponentially decreasing density in the gravity field, the operator \( A \) is acting on 4-dimensional vector:

\[
A = \begin{pmatrix}
0 & 0 \\
0 & \omega_1 \\
\alpha v_i & 0 \\
\beta v_i & 0
\end{pmatrix}, \quad n = 4
\]

and is the density of the fluid.

For the model which counts with salinity and heat transfer, the operator \( A \) acts on 5-dimensional vector:

\[
A = \begin{pmatrix}
0 \\
0 \\
0 & 0 \\
0 & \beta v_i \\
-\alpha v_i & \beta v_i
\end{pmatrix}, \quad n = 5
\]

In the considered problems, almost always, without loss of generality, the operator \( A \) may be considered as a skew-symmetric matrix.

We would like to analyze and to compare the results of the following two problems.

First, we consider the Cauchy problem for the system (1) for the case when the right-hand side \( \mathbf{F}(x,t) \) depends on time harmonically, i.e., \( \mathbf{F}(x,t) = \tilde{F}(x)e^{i\omega t}, \quad \omega \geq 0 \).

From the physical point of view, for this case, the solution of the Cauchy problem should stabilize and describe the mode of the induced vibrations with frequency \( \omega \).

Secondly, we consider the following problem of the normal...
vibrations for homogeneous system (1) associated with the Dirichlet boundary conditions $\frac{\partial v}{\partial \Omega} = 0 : v(x, t) = u(x) e^{-\omega t}$.

In this way, the term $\frac{\partial v}{\partial t}$ transforms into $-\lambda$, which permits the representation of the homogeneous system (1) as $M - \lambda I = 0$.

Thus, we obtain a spectral problem for the operator $M$.

We would like to compare the spectrum of the operator $M$ and the uniqueness of the limit amplitude of the induced vibrations for the Cauchy problem.

**Results and discussion**

For the Cauchy problem, in [1] there was proved the following result for the rotational motion.

**Theorem 1:** Let the exterior mass force be a periodic function with frequency $\omega \geq 0$:

$$\tilde{F}(x, t) = \tilde{f}(x)e^{-\omega t}.$$  

Then, the solution $v(x, t)$ of the Cauchy problem is also periodic and the stabilization property is valid:

$$\lim_{t \to \infty} v(x, t) = \tilde{U}(x),$$  

where $\tilde{U}(x)$ is the solution of the stationary system with the external force.

We observe that for the case of the stratified fluid, the Theorem 1 was proved in [2].

Let us recall that, for a linear self-adjoint operator $M$ acting in a Hilbert space, the essential spectrum is defined as the set of points of the continuous spectrum, limit points of the point spectrum, and eigenvalues of infinite multiplicity [3]. Therefore, the spectral points outside of the essential spectrum may only be eigenvalues of finite multiplicity.

For various types of the operator $A$, in [4,5] the following result was proved.

**Theorem 2:** There exists a finite constant, which is uniquely determined by the matrix, such that for the problem of normal vibrations, the essential spectrum of the operator is the interval of the imaginary axis:

$$\sigma_{\text{ess}}(M) = [-iB, iB].$$

Now, we would like to compare the relation between the uniqueness of the above limit amplitude of the induced vibrations of frequency, and the structure and localization of the essential spectrum of the normal vibrations.

The following result was proved in [6] for the rotational model, and later in [2] it was proved for the model of stratified fluid.

**Theorem 3:** If the frequency $\omega$ of the exterior mass forces does not belong to the essential spectrum of the normal vibrations, then the limit amplitude $\tilde{U}(x)$ is unique. On the other hand, if, however, the frequency $\omega$ belongs to the essential spectrum $\sigma_{\text{ess}}(M) = [-iB, iB]$, then the limit amplitude loses the uniqueness. From the physical point of view, it means that the resonance effect takes place.

We also would like to observe that, particularly, the explicit example of non-uniqueness of the solution may be found in [4] for the construction of the Weyl sequence [7] in the proof of the result for the essential spectrum.

**Conclusion**

The remarkable relation between the essential spectrum of normal vibrations and non-uniqueness of the limit amplitude may explain the effect of resonance of internal waves from a new mathematical point of view. May we also express a conjecture that similar results are valid for more general types of the operator $A$.

**References**


How to cite this article: Andrei G. Essential Spectrum of Internal Waves in the Ocean and Atmosphere: the Relation between the Frequency of the Induced Vibrations and Non-Uniqueness of the Limit Amplitude. Civil Eng Res J. 2018; 3(3): 555612. DOI: 10.19080/CERJ.2018.03.555612