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A Note about the Upgraded of Gauss-Markov Processes on a Banach Space: The Covariance Operator

Orchidea Maria Lecian*

Sapienza University of Rome, Rome, Italy.

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***Corresponding author:** Orchidea Maria Lecian, Sapienza University of Rome, Rome, Italy.

Abstract

The well-posed-ness of the covariance operator is newly proven for the Banach spaces achieved after upgrading the variance from the topological space of the σ-algebra of the Borel subset on a measurable space. The examples of a Hilbert space are analyzed. The kernels which define the convolution products of Gauss-Markov processes are newly proven to be defined after the covariance. The analytical definition of first-passage times id therefore now newly possible. The corresponding probability spaces are newly proposed. The new application to Cesaro averages is presented.

Keywords: Gauss-Markov processes; variance; covariance; first-passage times; probability space.

Introduction

In the work of Kaniuth et al. [1], the semi-simple Banach algebras and the regular, commutative algebras endowed with bounded, approximated identity are studied. Ibidem, the investigation is carried for the multipliers of the algebras, and of its power-bounded multipliers. The Gelfand spectrum of the algebras isconsidered.

In the work of Deriennic et al. [2], adapted, strictly aperiodic measures is a closed, proper subgroup are considered, in a uniformly convex Banach space. The ’average operator’ P is considered, and the convergence properties are studied. The convergence of the iterates of the ’average operator’ P^n is proven to be the weak one, which is upgraded to the strong one under suitable hypothesis. The tools ibidem developed are therefore apt to bring the investigation of Markov chains. In the work of Deriennic et al. [2], adapted, strictly aperiodic measures is a closed, proper subgroup are considered, in a uniformly convex Banach space.

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In the work of Mustafayev [3], the adapted strictly aperiodic measure μ is considered to calculate the iterates

T_μ^n of T_μ with T the bounded, continuous representation of a group $\mathcal{G}$ on a Banach space X. X is assumed to be uniformly convex and the measure μ is taken as adapted and strictly aperiodic. The aims of the present work are the study of the convolution kernels of Gauss-Markov processes. Such kernels are newly found to be defined after the covariance operator. In the work of Jones et al., probability spaces are considered, for which ergodic theorems for the convolution of the measure of a group are presented.

The paper is organized as follows.

- In Section 1, the definitions of measures are recalled, and the Gauss-Markov processes are reviewed.
- In Section 2, the convergence of the iterates of the convolution operators is studied.
- In Section 3, the convergence of the composition of the convolution operation of Markov kernels is studied.
- In Section 4, composition of the convolutions is newly

studied; the results are newly applied to Gauss-Markov kernels in the orthogonal-basis representation.

- In Section 5, the conclusion is proposed.

Introductory material

1.1. Definitions of measures

From the work of Deriennic et al [2], the investigation about measures is performed. Ibidem, the locally-compact group $\mathcal{J}$ is considered. Two regular probabilities measures ρ, υ o ℐ are taken.

The convolution product

$$\int f(t)dv * \rho(t) = \int f(u,v)dv(u)d\rho(v) \qquad (1)$$

is studied, where the supports of the measures are denominated and S_v and S_ρ ; respectively.

The following Propositions are recalled.

Proposition A:

The measure μ on a set S is irreducible if the closed subgroup generated after S is ℐ. □

Proposition B:

The measure μ o a set S is adapted if the closed semigroup generated after S in ℐ is $\bigcup_{n=1}^{n=\infty} S^n$. □

Proposition C:

The measure μ is strictly aperiodic is the smallest closed normal subgroup 𝓗(S) is a class containing the set S in ℐ. □

In the work of Mustafayev [3], the Banach space *X* is assumed to be uniformly convex, and the measure *μ* is taken as adapted and strictly aperiodic. The iterate μ^n is hypothesized to be non-singular with respect to the Haar measure on 𝓖.

From [2], the sequence T_μ^n is proven to converge strongly $\forall x \in X$

Form [4], the norm convergence and the almost-everywhere convergence of the iterates defined after T_μ in a probability space of *p* integration $L^p(\Omega, \Sigma, m)$ is taken, where the action of the group on the probability space $L^p(\Omega, \Sigma, m)$ is taken.

It is my aim to define the convolutions C_{uv} from Eq. (1), where

$$C_{uv} = \int f(uv)vdu\rho dv \qquad (2)$$

after kernels.

The result applies to Gauss Markov processes of aperiodic Markov chains.

1.2. Gauss-Markov processes

Real, continuous Gauss-Markov processes are schematized as $\{X(t), t \in \mathrm{T}\}$. In the work of Mehr et al. [3], the Gauss-Markov processes are characterized:

Definition a:

A Gauss-Markov process is one with covariance c (s, u) which obeys the factorization rule

$$c(s,t) = \frac{c(s,t)c(t,u)}{c(t,t)}, \qquad (3)$$

With

$$c(s,t) = h_1(s)h_2(t), \quad s \le t, \qquad (4a)$$

$$r(t) \equiv \frac{h_1(t)}{h_2(t)} > 0 \qquad (4b)$$

being the ration *r(t)* a monotonically-increasing function. Furthermore, the following properties are obeyed:

Definition b:

In a Gauss-Markov process, the following properties are obeyed:

I. *The continuous expectation value is defined*

$$\mathrm{E}[X(t)] = m(t); \qquad (5)$$

II. *The covariance c (s, t) is written as*

$$c(s,t) = \mathrm{E}[(X(s)-m(s))(X(t)-m(t))], \qquad (6)$$

which is continuous in $\mathrm{T} \times \mathrm{T}$;

III. *X(t) is non-singular at the endpoints, which qualify the interval of the* T *variable as*

$$\mathrm{T} \in [a,b]. \qquad (7)$$

A corollary of Definition b III) is enunciated as follows:

Corollary:

At the endpoints

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$$c(t,t) = \mathrm{E}[(X(t) - m(t))^2].$$

i.e. at t = a or at t = b. □

The convergence of the iterates of the convolution operators

In the work of Mustafayev [3], the convergence of the iterates of convolution operators from an Abelian group is proven. It is our aim to recall the main procedure in order for it to be generalized in the requested case. In [3], the locally compact Abelian group $\mathcal{G}$ is taken, with measure algebra M($\mathcal{G}$)

The measure μ is considered, with

$$\mu \in M(\mathcal{G}) \qquad (9)$$

The following Definition is here recalled, for it to be used in the discussion of the contractive mapping.

Definition c:

The measure μ is power-bounded as

$$sup_{n} \geq 0 \| \mu^{n} 1 < \infty. \qquad (10)$$

The following Theorem holds about t_g the iterate.

Theorem 1:

Let: $T = \{T_g : g \in \mathcal{G}$ be a bounded and continuous representation of $\mathcal{G}$ on X a Banach space.

$\forall_{\mu} M(G) \exists T_{\mu}$ a bounded linear operator on X associated with μ. □

1.3. **The algebra of all bounded linear operators**

Let *X* be a Banach space.

Let *B(X)* be the algebra of all bounded linear operators.

Let $\mathcal{G}$ be a locally-compact group.

Let μ be an arbitrary finite Borel measure $\mu \in \mathcal{G}$.

Given *T* a bounded and continuous representation of $\mathcal{G}$, the following Theorem holds

Theorem 2:

The measure μ integrates t_g.

It is here recalled that

$$\mu \equiv \mu(x) \qquad (11)$$

is needed in the developments.

Moreover,

Theorem 3:

μ^{n} is not singular with respect to the Haar measure on $\mathcal{G}$ for some $n \in \mathrm{N}$. □

1.4. **Progresses about the convergence**

The strong convergence of $\mathrm{T}_{\mu}^{n} x \forall x \in X$ is proven in [2].

The convergence in norm almost everywhere is provided with in [4]. Indeed, in the work of Deriennic et al. [2], some convergence properties of iterates of averages of objects are studied. The following Definitions are needed for our purposes, which are here recalled.

A locally-compact σ-compact group $\mathcal{Q}$ is for the moment considered.

The adapted, strictly aperiodic measure is chosen.

From the work of Deriennic et al. [2], the following Theorem is taken:

Theorem 4:

Let $\mathcal{Q}$ be a locally-compact σ-compact group; and let Y(t) be a continuous representation of $\mathcal{Q}$ after isometries in a uniformly convex Banach space $\mathcal{H}$. Furthermore, let μ be the opportune adapted.

When the average operator P is defined as

$$Px = \int \mathrm{T} x d\mu(x), \qquad (12)$$

then P^n converges strongly $\forall x \in \mathcal{H}$. □

The convergence properties when $\mathcal{Q}$ is Abelian or compact are ensured. It is here recalled that the Cesaro averages are taken as from the following

Theorem 5:

$$\frac{1}{n} \sum_{m=0}^{m=n=1} p^{m} x \qquad (13)$$

converge strongly $\forall x \in \mathcal{H}$ if $\mathcal{H}$ is (i) reflexive and (ii) if $\mathcal{H}$ is 'mean ergodic'. □

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Furthermore,

Theorem 6:

for every $\mathcal{H}$ Banach space, the sequence Eq. (13) is at least 'almost invariant'. □

Moreover, from the work of Deriennic et al. [2], the following Theorem is recalled:

Theorem 7:

Let Q be non-compact, and let μ be as from Theorem 4; therefore, $\forall f$ continuous, f vanishing at infinity,

$$\| \mu^n f \| \to 0 . \qquad (14)$$

□

About the convergence of the composition of the convolution operation of Markov kernels

The following new Theorem holds, which specifies the results from Section 3. The study of the convergence in norm almost everywhere is here redundant, as, for Markov processes, the iterates of the convolution operators are on a compact support.

Let *T* be a bounded a continuous representation of $\mathcal{G}$. Therefore, after the use of the contractive property, the following new Theorem holds:

Theorem 8:

Let T be a bounded and continuous representation of $\mathcal{G}$. Therefore, the measure is bounded as one with l^1 norm as

$$\sup \| \mu \| < \infty . \qquad (15)$$

Proof 8:

After the use of the contractive property. □

The following new Theorem is proven

Theorem 9:

et t_g be the iterate of the operator. Therefore, for a Markov chain, the convergence of the iteration with respect to a measure is ensured.

Proof 9:

The Markov chain iterates the convolution procedures. □

The further new Theorem holds

Theorem 10:

The measure μ integrates t_g. □

The study of Gauss-Markov processes is achieved as for the following

Theorem 11:

For Gauss-Markov processes, the convergence in norm is ensured after the definition of the iterate t_μ.□

The stationary Gauss Markov variance is therefore ensured as usual as

$$\mathrm{E}(x^2(t)) = \sigma^2 . \qquad (16)$$

In the work of Kaniuth et al [1], semi simple, regular, commutative Banach algebras are studied.

In the work of Jones et al. [4], ergodic theorems for Haussdorf, locally-compact groups are ensured. The almost-everywhere behavior(s) of the iterates of t_μ are investigated.

The results are extended to general locally-compact groups. Ibidem, convolutions of measure groups are studied. The measures are found for diverse measure spaces, and the representations are as unitary operators.

Gauss-Markov kernels in the orthogonal-basis representation: composition of the convolution

In the work of Mustafayev [3], the norm convergence and the almost-everywhere convergence of the iterates of bounded, continuous representation of a group on a Banach space are studied, int he cases the group is locally-compact. In the work of Mehr et al. [5], The Gauss-Markov processes are analyzed according to the covariance: the following new developments are therefore possible. The result from [5] is here used, which is recalled as follows.

Theorem 12: *The covariance $R_X(t_1, t_2)$ of a process $X(t): t \in \mathrm{T}$ is triangular if*

$$R_X(t_1, t_2) = F(t_1)G(t_2), \qquad (17)$$

With $t_2 \geq t_1, t_1, t_2 \in \mathrm{T}$.□

The following Theorem is recalled from the work of Doob [6]:

Theorem 13:

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A Gaussian process $\{\chi(t): t \in \mathrm{T}\}$ *is Markov iff it is endowed with triangular covariance.* □

I now specify the Theorem 12 and Theorem 13 as step processes at $t_1 = t_2$. I now recall that

Definition d:

In a Gauss-Markov process, the ordinary least square is characterized as one with the lowest (sampling) variance- the covariance being defined.

Let K_i be a collection of Markov Kernels, i.e., the collection of convolution Kernels from a Gauss-Markov process.

The measure is discussed as

Let μK be the measure for K; and let us consider the composition of two of them as $K_2 K_1$.

Here, one uses the left action of K^* the adjoint of K.

The composition is spelled as

$$(\mu K_2 K_1)(A) = ((\mu K_1)K_1)A. \quad (18)$$

The expression in Eq. (18) is proven to admit the representation

$$(\mu K_2 K_1)(A) = \int_X \int_X K_2(y, A) K_1(x, dy) d\mu(x). \quad (19)$$

The orthonormal basis is expressed as an orthogonal set of functions. I therefore define the new representation as one with Eq. (19) is therefore written as

$$\mu K_2 K_1 K = \int_X \int_X \kappa_2^* \kappa_1^* \mu dx \quad (20)$$

From the above, the covariance operator is written as from the current specification of Theorem 12, as ensured from the orthogonality properties of the basis functions (Gauss-Markov 'committors') in the new representation.

As a result, the following new definition is provided with

Definition e:

The Gauss-Markov orthogonal-basis representation is one in which the kernels are identified with the ratio function as

$$K(t) = R(t,t) \quad (21)$$

from Eq. (17).

The numerical calculation of the first-passage times of Gauss-Markov processes was performed in [7]. The first-passage times can now be calculated analytically. The corresponding probability space is defined after the tools provided in [4].

Conclusion

In the work of D'onofrio [8], a time-inhomogeneous Gauss-Markov process is used to modellize the solution of a stochastic differential equation, whose potential depends on time-depended forces and on an external load, on a two-dimensional strip; ibidem, the trajectories solutions of the differential equation are numerically simulated after a Euler-discretization algorithm: the boundaries of the domain of existence of the solution (i.e. the strip) are taken as 'absorbing states'. The covariance is fixed after the consideration of the potential of the differential equation. The first-exit times (FET's) are numerically estimated. The results find application in the study of protein dynamics. I have provided with the tools to calculate the FET's analytically.

In the work of Jones et al. [4], probability spaces are considered; furthermore, the point-wise convergence and the norm convergence are studied: the probability spaces of the Gauss-Markov processes are in the present work defined.

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