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Revisiting Banach's Matchbox Problem

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Abstract

We revisit the famous Banach's matchbox problem, where we give a fully probabilistic argument that probabilities arising in this context indeed sum up to one, that is, they define a probability mass function. Our proof avoids any combinatorial or inductive argument, and rather relies on probabilistic logic which students in the undergraduate level learning probability may find interesting.

Keywords: Banach's matchbox; Probability; Mass function; Combinatorial; Inductive argument; Probabilistic logic; Interesting; X matchsticks; Induction; Random variable; One box; Empty; Equal probability; Non-trivial combinatorial; Probabilistic logic; Bernoulli trials; Binomial random variable; Probability distribution; Symmetric nature; Equivalent

Introduction

Banach's matchbox problem is a widely known probability problem, discussed in depth in Feller [1], and may be found in numerous undergraduate probability textbooks. Feller attributes this problem to a humorous reference to the Polish mathematician Stefan Banach's smoking habits, in an address honoring him by H. Steinhaus. The problem goes like this: A certain mathematician carries two matchboxes each containing N matchsticks in his two pockets. Every time he needs to smoke, the mathematician chooses one of the two pockets at random. Suppose at some point of time, he finds that a box is empty. The problem is to find the probability that the other box contains exactly x matchsticks at that moment.

The solution to this problem is well-known and the proof can be found in many textbooks, including Feller [1]. In this note, we give a fully probabilistic argument to prove that the probabilities indeed define a probability mass function. In this regard, it is worth noting that Feller mentions that the above fact is not obvious analytically, and may be verified by induction on N. From that point of view, this proof, which argues solely from a probabilistic viewpoint, may prove to be of great interest to students learning undergraduate level probability.

Main result

For the sake of completeness, we first give the solution to the actual problem and then proceed to the formal proof of our main result. Let X be the random variable denoting the number of matchsticks in one box when the other box is found empty. Obviously, X can take the values $x=0,1,\ldots,N.$ We need to find the probabilities $Pr(X=x),\ x=0,1,\ldots,N.$ Now, the event that there are x matchsticks in one box when the other is found empty can be translated to the event of one box being chosen $(N+1)$ times while the other being chosen $(N-x)$ times. The latter event, for one fixed box (say, box on left pocket has been chosen $N+1$ times) has the probability given by $\binom{2N-x}{N}2^{-(2N-x+1)}$. Since the mathematician chooses both the boxes independently with equal probability, hence the required probability is given by

$$p(x)=P(X=x)=\binom{2N-x}{N}2^{-(2N-x)},x=0,1,\ldots,N. \qquad (1)$$

Since $p(x)$ define probabilities of a mutually exclusive and exhaustive set of events for $x=0,1,\ldots,N,$ hence they must sum up to unity. As mentioned earlier, if one considers the mathematical form of the probabilities and tries to show that they add up to unity, it will turn out to be a non-trivial combinatorial exercise. We prove the same using solely some probabilistic logic.

Theorem 1

The probabilities defined by $p(x)$ in equation (1) above define a probability mass function.

Proof

The probabilities $p(x)$ for all $x=0,1,\ldots,N,$ are strictly positive. Hence, to prove that $p(x)$ defines a probability mass function, we need to show that $\sum_{x=0}^{N}p(x)=1.$

The sum of the probabilities can be written as

$$\sum_{x=0}^{N}\binom{2N-x}{N}2^{-(2N-x)}=\sum_{y=0}^{N}\binom{N+y}{y}2^{-(N+y)}=2\sum_{y=0}^{N}\binom{N+1+y-1}{y}2^{-(N+y+1)}=2P(Y\le N), \qquad (2)$$

where Y denotes the number of failures preceding the $(N+1)th$ success in a sequence of Bernoulli trials with success probability $1/2$. Now we see that, $Pr(Y\le N)$ equals $Pr(U\le 2N+1),$ where $U=Y+N+1,$ which denotes the number of trials required to produce

$(N+1)$ successes in a sequence of Bernoulli trials with success probability $1/2$.

We now connect the random variable U with a Binomial random variable $V \sim \text{Binomial}(N+1, 1/2)$. Note that, $\{U \le 2N+1\}$ refers to the event that the number of trials to produce $(N+1)$ successes in a sequence of Bernoulli trials is at most $(2N+1)$. This is equivalent to say that in a sequence of $(2N+1)$. Bernoulli trials with probability of success $1/2$, there are at least $(N+1)$ successes. Hence we have, $\Pr(U \le 2N+1) = Pr(V \ge N+1)$. Due to symmetric nature of the probability distribution of V, we have, $\Pr(V \ge N+1) = 1/2$. Substituting this in equation (2), we get our desired result.

References

1. William Feller (2008) An Introduction to Probability Theory and Its Applications. Volume 1, John Wiley & Sons, USA.

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