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On Seemingly Unrelated Regressions with Linear Restrictions

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Abstract

The paper shows that, in the system of two seemingly unrelated regressions with linear restrictions, there are two methods can be used to obtain the best restricted least square estimator for the parameter of interest which involves a matrix power series, and thus conclude the best restricted least-square estimator only has unique simpler form.

Keywords: Seemingly unrelated regressions; Linear restrictions; Covariance-adjusted; Regressions; Biostatistics and econometrics; Non-diagonal; Matrix; estimator; Covariance-adjusted; Algebra; Kronecker; Unbiased; Uniqueness; Equivalent; Unique simpler version; Restrictions regression model; Less covariance; Semi-definite matrix; Unrelated regressions

Abbreviations: SUR: Seemingly Unrelated Regressions; LSE: Least-Square Estimator; GLSE: Generalized Least-Square Estimator

Introduction

The system of seemingly unrelated regressions (SUR) introduced by Zellner [1] and later developed by many authors which include, for example, Zellner [2], Revankar [3], Swamy & Mehta [4], Schmidt [5], Wang [6], Liu [7], Bhattacharya [8], Velu & Richards [9], Wang [10] and so on. So far SUR has been widely used in the field of biostatistics and econometrics, etc. Consider the system of two seemingly unrelated regressions, denoted by

$$\begin{cases} y_1 = X_1\beta + u_1 \\ y_2 = X_2\gamma + u_2, \end{cases} \tag{1.1}$$

in which $y_i\,(i=1,2)$ are $n\times 1$ observation vectors; $X_i\,(i=1,2)$ are $n\times p_i$ matrices with full column rank; β and γ are vectors of unknown regression parameters; u_1 and u_2 are correlated error vectors with mean zero and variance-covariance matrices $\mathrm{Cov}(u_i,u_j)=\sigma_{ij}I_n$, where $\sum(\sigma_{ij})$ is a 2×2 non-diagonal positive definite matrix. Suppose that it is known that the regression parameter β satisfies linear restrictions

$$L\beta = d, \tag{1.2}$$

where, without loss of generality, it is assumed that L is a given $q\times p_1$ matrix with full row rank and d is a given $q\times 1$ vector. Combining $L\beta = d$ with the first equation of (1.1), the well-known restricted least-square estimator (LSE) for β would be

$$\hat{\beta}_L = \left(I_{p1} - (X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}L\right)\hat{\beta} + (X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}d \tag{1.3}$$

With $\hat{\beta}_L = (X_1'X_1)^{-1}X_1'y_1$ [11].

Obviously, $\hat{\beta}_L$ only contains the information of the first equation but it does not make most use of all information of the system since $\sigma_{12}\neq 0$.

In the followings, under the condition that $\sum$ is known, firstly, we employ the covariance adjusted technique to improve $\hat{\beta}_L$ and step by step we come to the best restricted LSE for the parameter β Secondly, we show that the best restricted LSE of β can be obtained by integrating the generalized least-square estimator (GLSE) of β with $L\beta = d$. Finally, we prove an interesting fact which shows that the best restricted LSE only has unique simpler form, which allows us to study it in more details.

The best restricted LSE

Covariance-adjusted method: First, we state the covariance-adjusted lemma which introduced by Rao [12]and later developed by Wang [6] and Baksalary [13], etc.

Lemma 1: Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be $k_1\times 1$ and $k_2\times 1$ statistics with $E[\hat{\theta}_1]=\theta$ and $E[\hat{\theta}_2]=0$, where θ is an unknown parameter vector. Denote

$$\mathrm{Cov}\begin{pmatrix}\hat{\theta}_1\\ \hat{\theta}_2\end{pmatrix} = \begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}.$$

If $V_{12}\neq 0$, then there exists an unbiased estimator $\hat{\theta}=\hat{\theta}_1 - V_{12}V_{22}^{-}\hat{\theta}_2$ over a class of estimators $\mathrm{D}=\left\{T_1\hat{\theta}_1 + T_2\hat{\theta}_2 \,|\, T_1, T_2 \text{ are nonrandom matrices}\right\}$, and

$$\mathrm{Cov}\left(\hat{\theta}\right) = V_{11} - V_{12} - V_{22}^{-}V_{21} \le V_{11} = \mathrm{Cov}\left(\hat{\theta}_1\right),$$

Where V_{22}^{-} denotes any a generalized inverse of V_{22} and $A\ge B$ denotes $A - B$ is a real positive semi-definite matrix.

Proof: Omitted.

Wang [6] applies Lemma 1 to the system of m SURs and obtains a covariance adjusted estimator of regression parameter and discusses its properties. From Lemma 1, we find that as two unbiased estimators of θ, obviously, $\hat{\theta}$ is more efficient than $\hat{\theta}_1$ in the sense of having less covariance.

Hence, in order to obtain the most efficient restricted LSE of β, we need to find the most efficient estimator for β and use it to replace $\hat{\beta}$ in $\hat{\beta}_L$ By virtue of Lemma 1 and noting that $E[\hat{\beta}]=\beta$ and $E[N_2y_2]=0$, we firstly use N_2y_2 to improve $\hat{\beta}$ inside $\hat{\beta}_L$ and obtain a more efficient estimator $\hat{\beta}(1)$. Secondly we adjust $\hat{\beta}(1)$. by (Since N_1y_1 $E[N_1y_1]=0$) and obtain $\hat{\beta}(2)$, where $N_i=I_n-X_i(X_i'X_i)^{-1}X_i'$. Repeating this process, we come to the following estimator sequence for the parameter β:

$$\hat{\beta}(2k-1)=\hat{\beta}(2k-2)-\text{Cov}\left(\hat{\beta}(2k-2),N_2y_2\right)\left[\text{Cov}(N_2y_2,N_2y_2)\right]^{-}N_2y_2, \quad (1.4)$$

$$\hat{\beta}(2k)=\hat{\beta}(2k-1)-\text{Cov}\left(\hat{\beta}(2k-1),N_1y_1\right)\left[\text{Cov}(N_1y_1,N_1y_1)\right]^{-}N_1y_1, \quad (1.5)$$

$$k=1,2,\ldots,$$

Where $\hat{\beta}(0)=\hat{\beta}$.

After performing some algebra, we know that for $k\geq 1$

$$\hat{\beta}(2k-1)=(X_1'X_1)^{-1}X_1'\left(P_1+\rho^2N_2N_1\right)^{k-1}\left(y_1-\frac{\sigma_{12}}{\sigma_{22}}N_2y_2\right), \quad (1.6)$$

$$\hat{\beta}(2k)=(X_1'X_1)^{-1}X_1'\left(P_1+\rho^2N_2N_1\right)^{k}y_1 -\frac{\sigma_{12}}{\sigma_{22}}(X_1'X_1)^{-1}X_1'\left(P_1+\rho^2N_2N_1\right)^{k-1}N_2y_2, \quad (1.7)$$

Where $\rho^2=\sigma_{12}\sigma_{21}/(\sigma_{11}\sigma_{22})$ and $P_1=X_1(X_1'X_1)^{-1}X_1'$.

Thus, we obtain the restricted LSE sequence for β as follows

$$\hat{\beta}_L(2k-1)=\left(I_{p1}-(X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}L\right)\hat{\beta}(2k-1) +(X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}d,\ k\geq 1, \quad (1.8)$$

$$\hat{\beta}_L(2k)=\left(I_{p1}-(X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}L\right)\hat{\beta}(2) +(X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}d,\ k\geq 1. \quad (1.9)$$

Then, we have the following theorem.

Theorem 1. Let $\hat{\beta}_{L,best}$ denote the best restricted LSE of β. Then

$$\hat{\beta}_{L,best}=\lim_{k\to\infty}\hat{\beta}_L(2k-1)=\lim_{k\to\infty}\hat{\beta}_L(2k)$$

$$=\left(I_{p1}-(X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}L\right)$$

$$\times(X_1'X_1)^{-1}X_1'\left[I_n-\rho^2\sum_{i=0}^{\infty}\left(\rho^2P_2P_1\right)^{1}P_2N_1\right]\left(y_1-\frac{\sigma_{12}}{\sigma_{22}}N_2y_2\right)$$

$$+(X_1'X_1)^{-1}L'\left[L(X_1'X_1)^{-1}L'\right]^{-1}d,$$

Where $P_i=I_n-N_i$.

Proof: Theorem 1 follows from the following algebra fact

$$\left(X_1'X_1\right)^{-1}X_1'\left(P_1+\rho^2N_2N_1\right)^k$$

$$=\left(X_1'X_1\right)^{-1}X_1'\left[I_n+\rho^2N_2N_1+-\cdots+\left(\rho^2N_2N_1\right)^k\right]$$

$$=\left(X_1'X_1\right)^{-1}\left[X_1'-X_1'\cdot\rho^2P_2N_1-\cdots-X_1'\cdot\left(\rho^2P_2P_1\right)^{k-1}\cdot\rho^2P_2N_1\right]$$

$$=\left(X_1'X_1\right)^{-1}X_1'\left[I_n-\rho^2\sum_{i=0}^{\infty}\left(\rho^2P_2P_1\right)^{1}P_2N_1\right],\quad k=1,2\cdots. \quad (1.10)$$

The proof of Theorem 1 is finished.

GLSE method

Denote $y=(y_1',y_2')$, $X=\text{diag}(X_1,X_2)$, $\alpha=(\beta',\gamma')'$, $u=(u_1'u_2')'$. We can represent the system (1.1) as the following regression model

$$y=X\alpha+u,\quad Eu=0,\quad \text{Cov}(u,u)=\sum\otimes I_n, \quad (1.11)$$

Where $\otimes$ denotes the Kronecker product operator.

Then, the GLSE of the parameter β, say $\bar{\beta}$, Then, the GLSE of the parameter

$$\begin{pmatrix}\bar{\beta}\\ \bar{\gamma}\end{pmatrix}=\left[X'\left(\Sigma^{-1}\otimes I_n\right)X\right]^{-1}X'\left(\Sigma^{-1}\otimes I_n\right)y, \quad (1.12)$$

Which is the best linear unbiased estimator (BLUE) of β if Σ is known.

Lemma 2: In the regressions (1.1), if Σ is known, then the GLSE

$$\bar{\beta}=(X_1'X_1)^{-1}X_1'\left[I_n-\rho^2\sum_{i=0}^{\infty}\left(\rho^2P_2P_1\right)^{i}P_2N_1\right]\left(y_1-\frac{\sigma_{12}}{\sigma_{22}}N_2y_2\right).$$

Proof: Set $\Sigma^{-1}\otimes I_n=\left(\Sigma^{ij}\right)_{2\times 2}$ and $\left[X'(\Sigma^{-1}\otimes I_n)X\right]^{-1}\triangleq(Q^{ij})_{2\times 2}$. From (1.12), we know

$$\bar{\beta}=\left(Q^{11}X_1'\Sigma^{11}+Q^{12}X_2'\Sigma^{21}\right)y_1+\left(Q^{11}X_1'\Sigma^{12}+Q^{12}X_2'\Sigma^{22}\right)y_2, \quad (1.13)$$

Where

$$\Sigma^{11}=\frac{1}{\sigma_{11}}\cdot\frac{1}{1-\rho^2}I_n;\quad \Sigma^{12}=\frac{\sigma_{12}}{\sigma_{11}\sigma_{22}}\cdot\frac{1}{1-\rho^2}I_n;$$

$$\Sigma^{21}=\frac{\sigma_{11}}{\sigma_{11}\sigma_{22}}\cdot\frac{1}{1-\rho^2}I_n;\quad \Sigma^{22}=\frac{1}{\sigma_{22}}\cdot\frac{1}{1-\rho^2}I_n.$$

Following from the inverse of partitioned matrix, we obtain

$$Q^{11}=\left[X_1'\Sigma^{11}X_1-X_1'\Sigma^{12}X_2\left(X_2'\Sigma^{22}X_2\right)^{-1}X_2'\Sigma^{21}X_1\right]^{-1}$$

$$=\left[\frac{1}{\sigma_{11}\left(1-\rho^2\right)}X_1'X_1-\frac{\rho^2}{\sigma_{11}\left(1-\rho^2\right)}X_1'P_2X_1\right]^{-1}$$

$$=\sigma_{11}\left(1-\rho^2\right)\left(X_1'X_1\right)^{-1}\left[I_{p1}-\rho^2X_1'P_2X_1\left(X_1'X_1\right)^{-1}\right]^{-1}$$

$$=\sigma_{11}\left(1-\rho^2\right)\left(X_1'X_1\right)^{-1}\sum_{i=0}^{\infty}\left[\rho^2X_1'P_2X_1\left(X_1'X_1\right)^{-1}\right]^{i}, \quad (1.14)$$

and correspondingly

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$$Q^{12} = \sigma_{12}\left(1-\rho^2\right)\left(X_1'X_1\right)^{-1} X_1'\sum_{i=0}^{\infty}\left[\rho^2 P_2 P_1\right]^i X_2\left(X_2'X_2\right)^{-1}. \qquad (1.15)$$

Substituting (1.14) and (1.15) into (1.13) and noting the fact that $P_2P_1N_2 = -P_2N_1N_2$, we conclude that Lemma 2 is true.

Lemma 2 has been proved.

Then, we can state the following conclusion.

Theorem 2

The best restricted LSE $\hat{\beta}_{L,best}$ can be obtained by combining the GLSE with the restrictions.

Proof: Following from the above discussions, the conclusion is obvious.

The simpler form of $\hat{\beta}_{L,best}$

Theorem 3 $\hat{\beta}_{L,best}$ only has unique simpler form which given by

$$\hat{\beta}_{L,best}^{simpler} = \left(I_{p1} - (X_1'X_1)^{-1} L'\left[(X_1'X_1)^{-1} L'\right]^{-1} L\right)(X_1'X_1)^{-1} X_1'\left(y_1 - \frac{\sigma_{12}}{\sigma_{22}} N_2 y_2\right)$$
$$+ (X_1'X_1)^{-1} L'\left[(X_1'X_1)^{-1} L'\right]^{-1} d,$$

Proof: Note that the expression of $\hat{\beta}_{L,best}$ we find it is enough to show that if one term in $X_1'\sum_{i=0}^{\infty}(P_2P_1)^i P_2N_1$ is zero then the infinite sum is actually zero. Obviously, if $X_1'P_2N_1 = 0$, then $X_1'\sum_{i=0}^{\infty}(P_2P_1)^i P_2N_1 = 0$. For any a fixed $i(i \ge 1)$, if $X_1'(P_2P_1)^i P_2N_1 = 0$, then, firstly we have

$$X_1'(P_2P_1)^j P_2N_1 = 0, \quad j = i+1, i+2, \cdots. \qquad (1.16)$$

Secondly, from $X_1'(P_2P_1)^i P_2N_1 = 0 (i \ge 1)$, we have

$$X_1'(P_2P_1)^i (I_n - N_2) N_1 = 0$$
$$\Rightarrow X_1'(P_2P_1)^i N_2N_1 = 0$$
$$\Rightarrow X_1'(P_2P_1)^{i-1} P_2(I_n - N_1)N_2N_1 = 0$$
$$\Rightarrow X_1'(P_2P_1)^{i-1} P_2N_1N_2N_1 = 0$$
$$\Rightarrow X_1'(P_2P_1)^{i-1} P_2N_1N_2N_1P_2(P_1P_2)^{i-1} X_1 = 0,$$
$$\Rightarrow X_1'(P_2P_1)^{i-1} P_2N_1N_2 = 0$$
$$\Rightarrow X_1'(P_2P_1)^{i-1} N_2N_1N_2 = 0$$
$$\Rightarrow X_1'(P_2P_1)^i N_2N_1 = 0$$
$$\Rightarrow X_1'(P_2P_1)^i P_2N_1 = 0. \qquad (1.17)$$

Hence, step by step we obtain

$$X_1'(P_2P_1)^j P_2N_1 = 0, \quad j = 0, i+1, i+2, \cdots, i-1. \qquad (1.18)$$

Together with the above discussions, for any a fixed $i(i \ge 0)$, if $X_1'(P_2P_1)^i P_2N_1 = 0$ then we have

$$X_1'\rho^2\sum_{i=0}^{\infty}\left(\rho^2 P_2P_1\right)^i P_2N_1 = 0, \qquad (1.19)$$

Which implies the uniqueness of $\hat{\beta}_{L,best}^{simpler}$.

The proof of Theorem 3 is complete.

Remark 1: Also, it is easy to prove that (1) $X_1'(P_2P_1)^i P_2N_1 = 0$ is equivalent to $X_1'(P_2P_1)^i P_2N_1N_2 = 0$ for any $i \ge 0$; (2) if one term in $X_1'\rho^2\sum_{i=0}^{\infty}\left(\rho^2P_2P_1\right)^i P_2N_1 = 0$, or $X_1'\rho^2\sum_{i=0}^{\infty}\left(\rho^2P_2P_1\right)^i P_2N_1N_2 = 0$ is zero, both infinite sums are actually zero.

Conclusion

This note shows that

(a) The covariance-adjusted method is efficient and available, and the best restricted LSE can be obtained by employing covariance-adjusted technique.

(b) The best restricted LSE has unique simpler version, which allows us to discuss its properties in more details, especially in the case that Σ is unknown. Also, we find that when there are no restrictions and Σ is unknown, the two-stage estimator of β has unique simpler form $\tilde{\beta}^{simpler}(S) = \left(X_1'X_1\right)^{-1} X_1'y_1 - \frac{s_{12}}{S_{22}}\left(X_1'X_1\right)^{-1} X_1'N_2y_2$, which is the same as that of Revankar [3] and obtained by Revankar [3] under the condition that X_2 is a proper subset of X_1, also some authors obtain it under the assumptions that $P_1P_2 = P_2P_1X_1'X_2 = 0$, where S and $s_{i2}(i=1,2)$ denote the estimators of Σ and $\sigma_{i2}(i=1,2)$, respectively.

(c) The simpler form $\hat{\beta}_{L,best}^{simpler}$. is just integrating the one-step covariance-adjusted estimator with the restrictions.

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