L$^2$-Boundedness of Integral Operators Involving $\genfrac{[}{]}{0pt}{}{3}{2}$

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**Abstract**

In this paper, we formulate the integral operators $M_{s}^\sigma$ involving hypergeometric functions $\genfrac{[}{]}{0pt}{}{3}{2}$ as kernel. We discuss that these operators are composition of Erdlyi-Kober fractional integral operators. We also discuss the boundedness of these integral operators in $L^2$.

**Keywords:** Fractional integral transform; Liouville and Kober fractional integrals; Hypergeometric functions; Integral transform with hypergeometric functions in the kernel

There have made numerous investigations pertaining to integral operators involving various hypergeometric functions $\genfrac{[}{]}{0pt}{}{2}{1}$ and the confluent hypergeometric functions $\genfrac{[}{]}{0pt}{}{1}{1}$ as kernel [1-5]. Many authors also discussed the boundedness of integral operators and used their mapping properties to derive inversion processes [6].

In this paper, we use the integral representation of hypergeometric functions [7]

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-s)^{c-b-1}(1-t)^{-a}dt$$

for formulating the integral operators of the following form

$$M_{s}^\sigma(f(x)) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-s)^{c-b-1}(1-t)^{-a}dt$$

where

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-s)^{c-b-1}(1-t)^{-a}dt$$

Here we start with a basic result that use later, see Karapetiants and Samko [8] and Okikiol

**Lemma 1**

Suppose that $\psi$ is a measurable and homogeneous function of degree $\sigma$ for all real numbers $t$ i.e. $\psi(hx,ht) = |h|^{-\sigma}\psi(x,t)$.

Let

$$\Psi(f)(x) = \int_R f(t)\psi(x,t)dt,$$

then

$$\psi(f): L^2(R) \rightarrow L^2(R).$$

Also as a consequence, we have the $L^2$-boundedness of generalized Erdlyi-Kober fractional integrals [10] as transcribed below.
Lemma 2
Let
\[ f^{x-b}(y)(t) - \frac{\Gamma(c-b)}{\Gamma(c)} \int_t^x (x-t)^{c-1} f(t)dt, \quad 0 < t < x < \infty. \]
If \( c-b > 0, \sigma > 0 \) then \( f^{x-c}(f) : L^2 \rightarrow L^2 \).

We now prove the boundedness of the following integral operators involving homogeneous functions as kernel. These integral operators are generalization of integral operators those are studied by Love [11] and Habibullah [12].

Lemma 3
Let \( G^{a,b}_b(f)(x) = x^{a+b-1} \int_0^x (1 + x^2 t^2)^{\sigma} f(t)dt, \quad (x) > 0. \)
If \( 2\sigma(b-a) < 1 < 2\sigma b, 0 < a < 1, \) \( G^{a,b}_b(f) : L^1 \rightarrow L^1. \)

Proof. Note that
\[ G^{a,b}_b(V(f)(x)) = x^{a+b-1} \int_0^x (x^2 t^2 + y^2) \frac{a}{\Gamma(c)} f(y)dy. \]
If \( 2\sigma(b-a) < 1 < 2\sigma b, 0 < a < 1, \) there exists a constant \( k1= k1(a,b) \) such that
\[ \| G^{a,b}_b(f) \|_1 = \| G^{a,b}_b(V^2(f)) \|_1 \leq k1 \| V^2(f) \|_1 = k1 \| f \|_2 \]
that proves \( G^{a,b}_b(f) : L^2 \rightarrow L^2. \)

By using Fubini’s theorem, we have the following lemma:

Lemma 4
If \( f, g \in L^1(R) \), then
\[ \int_0^x g(t)G^{a,b}_b(f)(t)dt = \int_0^x f(t)G^{a,b}_b(g)(t)dt, \]
where
\[ G^{a,b}_b(g)(x) = x^{a+b-1} \int_0^x (1 + x^2 t^2)^{\sigma} g(t)dt. \]

Lemma 5
For \( \sigma > 0, \) let
\[ u^\sigma_x(t) = (x^2 - t^2)^{\sigma-1}, \quad 0 < t < x < \infty \]
\[ = 0, t \geq x. \]

Then
\[ u^\sigma_x(t) = \frac{\Gamma(c-b)}{\Gamma(c)} \int_t^x (x^2-t^2)^{\sigma} f(t)dt. \]

Proof. After making some substitutions in the integral representation of \( F^2 \), we get the following integral
\[ \int_t^x (x^2-t^2)^{\sigma} f(t)dt = \frac{\Gamma(c-b)}{\Gamma(c)} \int_t^x (x^2-t^2)^{\sigma} f(t)dt. \]
By replacing \( u^\sigma_x(t) \) in place of \( g \) in Lemma 4, we obtain
\[ G^{a,b}_b(u^\sigma_x(t)) = t^{\sigma-1} \int_t^x (x^2-t^2)^{\sigma} f(t)dt. \]

The implies that
\[ G^{a,b}_b(u^\sigma_x(t)) = \frac{\Gamma(c-b)}{\Gamma(c)} \int_t^x (x^2-t^2)^{\sigma} f(t)dt. \]

Now, we formulate integral operators \( u^\sigma_x(t) \) involving hypergeometric functions of the type \( r^\sigma_x \) and then prove the boundedness of these integral operators in \( L^2 \).

Theorem 1
Let
\[ M^{a,b}_\sigma(f)(x) = \frac{\Gamma(c-b)}{\Gamma(c)} \int_0^x f(t)G^{a,b}_b([x^2-t^2])dt \]

If \( 2\sigma(b-a) < 1 < 2\sigma b, 0 < a < 1, \) \( c-b > 0, \) then
\[ M^{a,b}_\sigma(f) : L^2 \rightarrow L^2. \]

Proof. An application of Lemma 3 and Lemma 4 yields
\[ P^{\sigma-1}(G^{a,b}_b(f))(x) = \frac{\Gamma(c-b)}{\Gamma(c)} \int_0^x f(t)G^{a,b}_b([x^2-t^2])dt. \]

By using Lemma 5, we conclude that
\[ M^{a,b}_\sigma(f)(x) = \frac{\Gamma(c-b)}{\Gamma(c)} \int_0^x f(t)G^{a,b}_b((u^\sigma_x(t))dt. \]

Consequently, it implies that
\[ M^{a,b}_\sigma(f)(x) = M^{a,b}_\sigma(f)(G^{a,b}_b(f)(x)), \]
Since \( f^{x-c-b}_b : L^2 \rightarrow L^2 \) by Lemma 2, it follows from Lemma 3 that if \( 2\sigma(b-a) < 1 < 2\sigma b, 0 < a < 1, \) \( c-b > 0, \) then
\[ \| M^{a,b}_\sigma(f) \|_2 \leq K \| f \|_2. \]

Hence,
\[ M^{a,b}_\sigma(f) : L^2 \rightarrow L^2. \]

References


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