Biostatistics and Biometrics
Open Access Journal
ISSN: 2573-2633

Juniper Publishers key to the Researchers

Perspective
Volume2 Issue 2 - July 2017
DOI: 10.19080/BBOAJ.2017.02.555583

Biostat Biometrics Open Acc J

L^2-Boundedness of Integral Operators Involving ${}_3F_2^{\sigma}$

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Submission: May 5, 2017; **Published:** July 11, 2017

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Abstract

In this paper, we formulate the integral operators $M_{\sigma,b}^{\sigma,a}$ involving hypergeometric functions ${}_3F_2^{\sigma}$ as kernel. We discuss that these operators are composition of Erdlyi-Kober fractional integral operators. We also discuss the boundedness of these integral operators in L^2.

Keywords: Fractional integral transform; Liouville and Kober frac-tional integrals; Hypergeometric functions; Integral transform with hy-pergeometric functions in the kernel

There have made numerous investigations pertaining to integral operators involving various hypergeometric functions ${}_2F_1$ and the confluent hypergeometric functions ${}_1F_1$ as kernel [1-5]. Many authors also discussed the boundedness of integral opera- tors and used their mapping properties to derive inversion processes [6].

In this paper, we use the integral representation of hypergeomet- ric functions [7]

$$ {}_3F_2\left[\begin{matrix} a,\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2}; \end{matrix} x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-x^2t)^{-a}\,dt $$

for formulating the integral operators of the following form

$$ M_{\sigma,b}^{\sigma,a}(f)(x) = \int_0^{\infty}(xt)^{\sigma b-1}\times {}_3F_2^{\sigma}\left[\begin{matrix} \frac{a}{2\sigma},\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2}; \end{matrix} -x^{2\sigma}t^{2\sigma}\right] f(t)\,dt, $$

where

$$ {}_3F_2^{\sigma}\left[\begin{matrix} \frac{a}{2\sigma},\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2}; \end{matrix} -x^{2\sigma}t^{2\sigma}\right] = \frac{\sigma\Gamma(c)x^{\sigma-\sigma c}}{\Gamma(b)\Gamma(c-b)}\int_0^x y^{\sigma b-1}(x^{\sigma}-y^{\sigma})^{c-b-1}(1+t^{2\sigma}y^{2\sigma})^{\frac{-a}{2\sigma}}\,dt. $$

Here we start with a basic result that use later, see Karapetiants and Samko [8] and Okikiol

Lemma 1

Suppose that ψ is a measurable and homogeneous function of degree $^{-1}$ for all real numbers h i.e. $\psi(h_x,h_t) = |h|^{-1}\psi(x,t)$.

Let

$$ \Psi(f)(x) = \int_R f(t)\psi(x,t)\,dt, $$

then

$$ \psi(f): L^2(R) \to L^2(R) \cdot $$

Also as a consequence, we have the L^2-boundedness of generalized Erdlyi-Kober fractional integrals [10] as transcribed below.

Lemma 2

Let

$$I^{\sigma,c-b}(f)(x)=\frac{\sigma x^{\sigma(b-c)+\sigma-1}}{\Gamma(c-b)}\int_0^x (x^\sigma - t^\sigma)^{c-b-1} f(t)dt,\quad 0 < t < x < \infty.$$

If c-b > 0, σ > 0 then . $I^{\sigma,c-b}(f): L^2 \to L^2$

We now prove the boundedness of the following integral operators involving homogeneous functions as kernel. These integral operators are generalization of integral operators those are studied by Love [11] and Habibullah [12].

Lemma 3

Let $G_b^{\sigma,a}(f)(x)=x^{\sigma b-1}\int_0^\infty t^{b-1}(1+x^{2\sigma}t^{2\sigma})^{\frac{-a}{2\sigma}} f(t)dt,$ (x>0).

If 2σ(b − a) < 1 < 2σb, 0 < a < 1, then $G_b^{\sigma,a}(f): L^2 \to L^2$.

Proof. Note that

$$G_b^{\sigma,a}(V(f))(x)=x^{\sigma b-1}\int_0^\infty y^{a-\sigma b}(x^{2\sigma}+y^{2\sigma})^{-\frac{a}{2\sigma}} f(y)dy.$$

If 2σ(b−a) < 1 < 2σb, 0 < a < 1, there exists a constant k1= k1(a,b) such that

$$\| G_b^{\sigma,a}(f)\|_2 = \| G_b^{\sigma,a}(V^2(f))\|_2 \le k_1 \| V^2(f)\|_2 = k_1 \| f \|_2$$

that proves $G_b^{\sigma,a}(f): L^2 \to L^2$.

By using Fubini's theorem, we have the following lemma:

Lemma 4

If $f, g \in L^2(R)$, then

$$\int_0^\infty g(t) G_b^{\sigma,a}(f)(t)dt = \int_0^\infty f(t) G_b^{*\sigma,a}(g)(t)dt,$$

where

$$G_b^{*\sigma,a}(g)(x)=x^{\sigma b-1}\int_0^\infty t^{\sigma b-1}(1+x^{2\sigma}t^{2\sigma})^{-\frac{a}{2\sigma}} g(t)dt.$$

Lemma 5

For σ > 0, let

$$u_x^\sigma(t)=(x^\sigma-t^\sigma)^{c-b-1},\quad 0<t<x<\infty$$
$$=0, t \ge x.$$

Then

$$G_b^{\sigma,a}(u_x^\sigma)(t)=\frac{\Gamma(b)\Gamma(c-b)}{\sigma\Gamma(c)} x^{\sigma c-\sigma} t^{\sigma b-1}\, {}_3F_2^\sigma\left[\begin{matrix}\frac{a}{2\sigma},\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2};\end{matrix} -x^{2\sigma}t^{2\sigma}\right]$$

Proof. After making some substitutions in the integral representation of 3F2, we get the following integral

$${}_3F_2^\sigma\left[\begin{matrix}\frac{a}{2\sigma},\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2};\end{matrix} -x^{2\sigma}t^{2\sigma}\right]=\frac{\sigma\Gamma(c)x^{\sigma-\sigma c}}{\Gamma(b)\Gamma(c-b)}\int_0^x y^{\sigma b-1}(x^\sigma-y^\sigma)^{c-b-1}(1+t^{2\sigma}y^{2\sigma})^{\frac{-a}{2\sigma}} dt.$$

By replacing u_x^σ in place of g in Lemma 4, we have obtain

$$G_b^{*\sigma,a}(u_x^\sigma)(t)=t^{\sigma b-1}\int_0^\infty y^{\sigma b-1}(x\sigma^\sigma-y^\sigma)^{c-b-1}(1+x^{2\sigma}y^{2\sigma})^{-\frac{a}{2\sigma}} dy.$$

The implies that

$$G_b^{*\sigma,a}(u_x^\sigma)(t)=\frac{\Gamma(b)\Gamma(c-b)}{\sigma\Gamma(c)} t^{\sigma c-1} x^{\sigma c-1}\, {}_3F_2^\sigma\left[\begin{matrix}\frac{a}{2\sigma},\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2};\end{matrix} -x^{2\sigma}t^{2\sigma}\right].$$

Now, we formulate integral operators $M_{\sigma,b}^{\sigma,a}$ involving hypergeometric functions of the type ${}_3F_2^\sigma$ and then prove the boundedness of these integral operators in L^2.

Theorem 1

Let

$$M_{\sigma,b}^{\sigma,a}(f)(x)=\int_0^\infty (xt)^{\sigma b-1} \times {}_3F_2^\sigma\left[\begin{matrix}\frac{a}{2\sigma},\frac{b}{2},\frac{b+1}{2}; \\ \frac{c}{2},\frac{c+1}{2};\end{matrix} -x^{2\sigma}t^{2\sigma}\right] f(t)dt.$$

If 2σ(b-a) < 1 < 2σb, 0 < a < 1, c-b > 0, then

$M_{\sigma,b}^{\sigma,a}(f)=CI^{\sigma,c-b}(G_b^{\sigma,a}(f)),$ where $C=\frac{\Gamma(c)}{\Gamma(b)}$ and

$$M_{\sigma,b}^{\sigma,a}(f): L^2 \to L^2.$$

Proof. An application of Lemma 3 and Lemma 4 yields

$$I^{\sigma,c-b}(G_b^{\sigma,a}(f))(x)=\frac{\sigma x^{\sigma(b-c)+\sigma-1}}{\Gamma(c-b)}\int_0^\infty f(t) G_b^{*\sigma,a}\left[(x^\sigma-t^\sigma)^{c-b-1}\right]dt$$

$$=\frac{\sigma x^{\sigma(b-c)+\sigma-1}}{\Gamma(c-b)}\int_0^\infty f(t) G_b^{*\sigma,a}(u_x^\sigma)(t)dt.$$

By using Lemma 5, we conclude that

$$M_{\sigma,b}^{\sigma,a}(f)(x)=\frac{\Gamma(c)x^{\sigma(b-c)+\sigma-1}}{\Gamma(b)\Gamma(c-b)}\int_0^\infty f(t) G_b^{*\sigma,a}(u_x^\sigma)(t)dt.$$

Consequently, it implies that

$M_{\sigma,b}^{\sigma,a}(f)(x)=AI^{\sigma,c-b}(G_b^{\sigma,a}(f))(x),$ where $C=\frac{\Gamma(c)}{\Gamma(b)}$.

Since $I^{\sigma,c-b}: L^2 \to L^2$ by Lemma 2, it follows from Lemma 3 that if 2σ(b- a) < 1 < 2σb, 0 < a < 1, c-b > 0, then

$$\| M_{\sigma,b}^{\sigma,a}(f)\|_2 \le K \| f \|_2 .$$

Hence,

$$M_{\sigma,b}^{\sigma,a}(f): L^2 \to L^2.$$

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How to cite this article: Shahid M. L²-Boundedness of Integral Operators Involving ${}_3F_2^\sigma$. Biostat Biometrics Open Acc J. 2017;2(2): 555583. DOI: 10.19080/BBOAJ.2017.02.555583.

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