Hybrid Parametric Optimality Constraints for Discrete Minmax Fractional Programming

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Abstract

Several classes of generalized higher order parametric sufficient optimality constraints for a discrete minmax fractional programming problem are investigated toward establishing advanced results on higher order fractional programming. These results are established by applying advanced partitioning schemes and various types of generalized second-order \((F,\beta,\phi,\rho,\theta,m)\)-univexity assumptions. The obtained results are new and generalize most of the results on \((F,\beta,\phi,\rho,\theta)\)-univexity in the literature.

Keywords: Discrete minmax fractional programming; Second-order univex functions; Generalized sufficient optimality conditions

Introduction

In this paper, we intend to establish several sets of generalized parametric sufficient optimality conditions for the following discrete minmax fractional programming problem:

\[
(P): \text{Minimize } \max_{1 \leq i \leq p} f_i(x)
\]

subject to \(G_j(x) \leq 0, \quad H_k(x) = 0, \quad K \in \mathbb{L}, \quad x \in X,
\]

where \(X\) is an open convex subset of \(\mathbb{R}^n\) \((n\text{-dimensional Euclidean space})\), \(f_i, g_j, 1 \leq i \leq p\), and \(H_k, K \in \mathbb{L}\), are real-valued functions defined on \(X\), and for each \(j \in q, g_j(x) > 0\) for all \(x\) satisfying the constraints of \((P)\). Let \(F\) denote the feasible set (assumed to be nonempty) for \((P)\) defined by

\[
F = \{x \in X : G_j(x) \leq 0, \quad j \in q, \quad H_k(x) = 0, \quad k \in \mathbb{L}\}.
\]

The present investigation is aimed at establishing various second-order necessary and sufficient optimality conditions for several types of optimization problems, using the generalized concepts of second-order invexity, pseudoinvexity, and quasiinvexity originally defined by Hanson [1], and a set of second-order necessary optimality conditions by introducing the new classes of generalized second-order invex functions. We shall apply two partitioning schemes by Mond & Weir [2] and Yang [3], in conjunction with the new classes of generalized second-order invex functions to formulate and discuss numerous sets of generalized second-order sufficient optimality conditions for \((P)\). To the best of our knowledge, all the second-order sufficient optimality results established in this paper are new in the area of discrete minmax fractional programming and encompass most of the investigations in the literature. The generalized optimality conditions established here can be utilized for constructing some generalized second-order parametric duality models for \((P)\) and proving numerous weak, strong, and strict converse duality theorems. For more details on the discrete minmax fractional programming and related literature, we refer the reader [1-9].

The rest of this paper is organized as follows. In the remainder of this section, we generalize a few basic definitions and recall some auxiliary results which will be needed in the sequel. In Section 2, we state and prove various second-order parametric sufficient optimality results for \((P)\) using a variety of generalized \((F,\beta,\pi,\phi,\rho,\theta,m)\)-sounivexity assumptions. Finally, in Section 3 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem investigated in the present paper.

We next define some new classes of generalized second-order univex functions, called (strictly) \((F,\beta,\phi,\pi,\rho,\theta,m)\)-sounivex, (strictly) \((F,\beta,\phi,\pi,\rho,\theta,m)\)-pseudosounivex, and (pre-strictly) \((F,\beta,\pi,\phi,\rho,\theta,m)\)-quasisounivex functions. These are further extensions of the classes of second-order (strictly) \((\phi,\eta,\rho,\theta,m)\)-sounivex, (strictly) \((\phi,\eta,\rho,\theta,m)\)-pseudosounivex, and (prestrictly) \((\phi,\eta,\rho,\theta,m)\)-quasisounivex functions which were introduced recently in [4]. The second-order univex functions are also referred to as “sounivex functions” in the litera-ture.
refer the reader [8,9]. Now we present the new classes of $(F,\beta,\pi,\varphi,\rho,\theta,\lambda)$-sounivex functions at $x^*$. Let $f : X \to \mathbb{R}$ be a twice differentiable function.

**Definition**

The function $f$ is said to be (strictly) $(F,\beta,\pi,\varphi,\rho,\theta,\lambda)$-sounivex at $x^*$ if there exist functions $\beta : X \times X \to \mathbb{R}, \pi : X \times X \to \mathbb{R}, \theta : X \times X \to \mathbb{R}^n$, a sublinear function $\phi(x,x^*), \mathbb{R}^n \to \mathbb{R}$, and a positive integer $m$ such that for each $z \in X(x \times x^*)$, and $z \in \mathbb{R}^n$.

$$\phi(f(x) - f(x^*)(x) \geq F(x,x^*) + \beta(x,x^*) \nabla f(x^*)$$

$$+ \left< \pi(x,x^*) \nabla^2 f(x^*) x \right>$$

$$- \frac{1}{2} \left( \lambda \nabla^2 f(x^*) z \right)$$

$$\geq - \rho(x,x^*) \| \theta(x,x^*) \|^m,$$

where $\| \cdot \|$ is a norm on $\mathbb{R}^n$ and $\langle a, b \rangle$ is the inner product of the vectors $a$ and $b$. The function $f$ is said to be (strictly) $(F,\beta,\pi,\varphi,\rho,\theta,\lambda)$-sounivex on $X$ if it is (strictly) $(F,\beta,\pi,\varphi,\rho,\theta,\lambda)$-sounivex at each $x^* \in X$.

**Definition**

The function $f$ is said to be (strictly) $(F,\beta,\pi,\varphi,\rho,\theta,\lambda)$-pseudosounivex at $x^*$ if there exist functions $\beta : X \times X \to \mathbb{R}, \pi : X \times X \to \mathbb{R}, \theta : X \times X \to \mathbb{R}^n$, Hybrid optimality constraints a sublinear function $F(x,x^*), \mathbb{R}^n \to \mathbb{R}$, and a positive integer $m$ such that for each $x \in X(x \times x^*)$, and $z \in \mathbb{R}^n$.

$$F(x,x^*) + \beta(x,x^*) \nabla f(x^*)$$

$$+ \left< \pi(x,x^*) \nabla^2 f(x^*) x \right>$$

$$- \frac{1}{2} \left( \lambda \nabla^2 f(x^*) z \right)$$

$$\geq - \rho(x,x^*) \| \theta(x,x^*) \|^m.$$
Second-Order Sufficient Optimality

In this section, we discuss several families of sufficient optimality results under various generalized $(F,\beta,\phi,\pi,\rho,\theta,m)$-souvinex hypotheses imposed on the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [2] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{I_\mu, J_\mu : \mu \in M \}$ and $(K_\mu, K_\mu, K_\mu, K_\mu)$ be partitions of the index sets $\mathbb{Q}$ and $\mathbb{Z}$, respectively; thus, $J_\mu \subseteq \mathbb{Q}$ for each $\mu \in M \cup \{0\}$, $J_\mu \cap J_\mu = \emptyset$ for each $\mu \neq \nu$, and $U_\mu \cup U_\mu = \mathbb{Q}$.

Obviously, similar properties hold for $(K_\mu, K_\mu, K_\mu, K_\mu)$.

Moreover, if $M_1$ and $M_2$ are the numbers of the partitioning sets of $\mathbb{Q}$ and $\mathbb{Z}$, respectively, then $M = \max\{M_1, M_2\}$ and $I_\mu = \emptyset$ or $K_\mu = \emptyset$ for $\mu > \min\{M_1, M_2\}$.

In addition, we use the real-valued functions $\xi \mapsto \Phi(\xi; u, v, w, \lambda)$, $\xi \mapsto \Phi(x; u, v, w, \lambda)$, and $\xi \mapsto \Lambda(\xi; u, v, w)$ defined, respectively, for $(\mathbb{Q}, \mathbb{Z})$.

In the proofs of our sufficiency theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma

For each $x, x \in \mathcal{X}$, [7]

$$
\phi(x) = \max_{i \in p} \varphi_i(x) = \max_{i \in p} \sum_{j \in \mathbb{Q}} h_{ij}(x)
$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized second-order parametric sufficient optimality results for (P) as follows.

Theorem

Let $x \in \mathcal{E}$, the $x^* = \phi(x)$, and assume that the functions $f_i$, $g_j$, $i \in p$, $j \in \mathbb{Q}$, and $h_{ik}, k \in \mathbb{R}$, are twice differentiable at $x^*$, and that for each there exist $x^* \in U$, $x^* \in \mathbb{R}$, and $w^* \in \mathbb{R}$, such that

$$
\frac{\partial}{\partial x_j} f_i(x^*) = 0, \quad j \in \mathbb{Q}.
$$

where $C(x^*)$ is the set of all critical directions of (P) at $x^*$, that is,

$$
C(x^*) = \{ z \in \mathbb{R} : \langle g_j(x^*) \rangle_{j=1}^J = 0, \quad i \in A(x^*) \}
$$

$$
\langle g_j(x^*) \rangle_{j=1}^J \leq 0, \quad j \in B(x^*), \langle h_k(x^*) \rangle_{k=1}^K \leq 0, \quad k \in L
$$

$A(x^*) = \{ j \in p : f_j(x^*) \neq 0, \quad g_j(x^*) = \max_{i \in \mathbb{Q}} f_i(x^*) / g_i(x^*) \}$, and $B(x^*) = \{ j \in p : g_j(x^*) = 0 \}.$

Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:

a. $\xi \mapsto \Phi(\xi; u, v, w, \lambda^*)$ is prestrictly $(F,\beta,\phi,\pi,\rho,\theta,m)$-pseudouivex at $x^*$ and $\phi(0) \geq 0$;

b. For each $\xi \mapsto \Lambda(\xi; v, w)$ is strictly $(F,\beta,\phi,\pi,\rho,\theta,m)$-pseudouivex at $x^*$, $\phi(0)$ is increasing, and $\phi(0) = 0$;

c. $\mathcal{P}(x^*) + \sum_{i=1}^M \phi_i(x, x^*) \geq 0$ for all $x \in \mathcal{F}$;

d. $\xi \mapsto \Phi(\xi; u, v, w, \lambda^*)$ is prestrictly $(F,\beta,\phi,\pi,\rho,\theta,m)$-pseudouivex at $x^*$ and $\phi(0)$ is a null set $\{a \} \geq 0$.

Then $x^*$ is an optimal solution of (P).
(2.1) and (2.2) can be expressed as follows:

$$F_{\{x^*, \beta(x^*)\}} = \sum_{i=1}^{M} \sum_{j=1}^{N_i} \beta_i \frac{\partial G_j(x^*)}{\partial x_i} + \sum_{i=1}^{M} \sum_{k=1}^{N_{ik}} \alpha_i \frac{\partial H_k(x^*)}{\partial x_i}$$

$$F_{\{x^*, \beta(x^*)\}} = \sum_{i=1}^{M} \sum_{j=1}^{N_i} \beta_i \frac{\partial G_j(x^*)}{\partial x_i} + \sum_{i=1}^{M} \sum_{k=1}^{N_{ik}} \alpha_i \frac{\partial H_k(x^*)}{\partial x_i}$$

(by the feasibility of $\rho \theta$), it follows from (ii)

$$\rho \theta (x, x^*) \geq 0, \quad \text{and hence we get}$$

$$\sum_{i=1}^{M} \sum_{j=1}^{N_i} \beta_i \frac{\partial G_j(x^*)}{\partial x_i} + \sum_{i=1}^{M} \sum_{k=1}^{N_{ik}} \alpha_i \frac{\partial H_k(x^*)}{\partial x_i} \geq 0$$

Since for each $t \in M$, $A_t(x, v, w) = \sum_{j=1}^{M} v^i G_j(x) + \sum_{i=1}^{M} w^i H_k(x^*)$

(by the feasibility of $x^*$)

$$A_t(x^*, y^*, w^*)$$

and hence $\tilde{\phi}(A_t(x^*, y^*, w^*) - A_t(x^*, y^*, w^*)) \leq 0$. It follows from (ii) that

$$\tilde{\phi}(A_t(x^*, y^*, w^*) - A_t(x^*, y^*, w^*)) \leq 0 \quad \text{by (2.4) and the feasibility of} x^*$$

which by virtue of (i) implies that

$$\Phi(x, u^*, v^*, w^*, \lambda^*) = \Phi(x, u^*, v^*, w^*, \lambda^*) \geq 0.$$
for each $x^* \in C(x^*)$, there exist $u^* \in U, v^* \in \mathbb{R}^+$, and $w^* \in \mathbb{R}^+$ such that (2.1)-(2.4) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:

\begin{enumerate}[a.]
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-quasisounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{a}
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-pseudosounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{b}
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-quasisounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{c}
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-pseudosounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{d}
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-quasisounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{e}
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-pseudosounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{f}
  \item For each $i \in I_+ \equiv \{i \in I: u_i^* > 0\}$, $\phi_i(x, v^*, w^*)$ is $(F, \beta, \tilde{\phi}_i, \pi, \rho_1, \theta, m)$-quasisounivex at $x^*$, $\phi_i$ is strictly increasing, and $\tilde{\phi}_i(0) = 0$; \label{g}
\end{enumerate}

Then $x^*$ is an optimal solution of (P).

\section*{Proof}

(a): Suppose to the contrary that $x^*$ is not an optimal solution of (P). Then there is a feasible solution $x' \neq x^*$ of (P) such that $\phi(x') < \phi(x^*) = \lambda^\ast$. Hence it follows that

\begin{equation} \tag{2.10}
  f_j(x) - \lambda^\ast g_j(x) < 0 \quad \text{for each } i \in P.
\end{equation}

Keeping in mind that $v^* \geq 0$, we see that for each $i \in I_+$, $\phi_i(x^*, v^*, \lambda^\ast) = f_j(x) - \lambda^\ast g_j(x) + \sum_{j \in J} w^*_j g_j(x) + \sum_{k \in K_0} w^*_k h_k(x)$ is not an optimal solution of (P). Then there is a feasible solution $x' \neq x^*$ of (P) such that $\phi(x') < \phi(x^*) = \lambda^\ast$. Hence it follows that

\begin{equation} \tag{2.11}
  f_j(x) - \lambda^\ast g_j(x) < 0 \quad \text{for each } i \in P.
\end{equation}

and so using the properties of the function $\tilde{\phi}_i$, we get

\begin{equation}
  \phi_i(x, v, \lambda) = \phi_i(x, v, w, \lambda)
\end{equation}

which in view of (i) implies that
\[ \nabla^2 f_j(x^*) - \lambda^2 V^2 G_j(x^*) + \sum_{t \in I} \nabla^2 V^2 H_t(x^*) \leq 0. \]

Since \( u^* > 0, u_t^* = 0 \) for each \( i \in I \setminus I_1 \), \( \sum_{i=1}^p u_t^* = 1 \) and \( F(x, x^*, \cdot) \) is sublinear, the above inequalities yield
\[
\sum_{i=1}^m \mu_i \nabla^2 V^2 G_j(x^*) + \sum_{t \in I} \nabla^2 V^2 H_t(x^*) \leq \sum_{i=1}^m \mu_i \nabla^2 V^2 G_j(x^*) + \sum_{t \in I} \nabla^2 V^2 H_t(x^*) \leq 0.
\]

Proceeding as in the proof of Theorem 2.1, we see that our assumptions in (ii) lead to
\[
\beta(x, x^*) \geq -\sum_{i=1}^m \mu_i \nabla^2 V^2 G_j(x^*) + \sum_{t \in I} \nabla^2 V^2 H_t(x^*) \geq 0.
\]

which when combined with (2.5) and (2.6) results in
\[
\frac{1}{2} \left( \sum_{i=1}^m \mu_i \nabla^2 f_j(x^*) - \lambda^2 V^2 G_j(x^*) + \sum_{t \in I} \nabla^2 V^2 H_t(x^*) \right) \leq -\sum_{i=1}^m \mu_i \nabla^2 f_j(x^*) + \sum_{t \in I} \nabla^2 V^2 H_t(x^*) \geq 0.
\]

In view of (iii), this inequality contradicts (2.11). Hence, \( x^* \) is an optimal solution of (P).

(b)-(g): The proofs are similar to that of part (a).

In the next theorem, we make use of a slightly different partitioning method which appears to have been used for the first time by Yang [3] for the purpose of formulating a general duality model for a multi objective fractional programming model, to present another collection of sufficient optimality results for (P) which are somewhat different from those stated in Theorems 2.1 and 2.2. These results are formulated by utilizing a partition of in addition to those of and , and by placing appropriate generalized \((F, \beta, \phi, \Pi, \rho, \theta, m)\) - sounivexity requirements on certain combinations of the problem functions.

Let \( \{I_0, I_1, \ldots, I_l\} \) be a partition of \( p \) such that \( I = \{0, 1, 2, \ldots, I\} = \mathcal{M} = \{\emptyset, 1, \ldots, \mathcal{M}\} \), and let the real-valued function \( \xi \rightarrow \prod_i (\xi, u, v, w, \lambda) \) be defined, for fixed \( u, v, w, \lambda \), by
\[
\Pi_i (\xi, u, v, w, \lambda) = \sum_{t \in I} [f_j(\xi) - \lambda g_j(\xi)] + \sum_{i=1}^m \nabla G_j(\xi) + \sum_{t \in I} \nabla H_t(\xi), t \in M.
\]

Theorem

Let \( x^* \in F \), let \( \lambda^* = (\phi(x^*), \phi) \), and assume that the functions \( f_j, g_i, i \in p, g_j, j \in q \), and \( H_k, k \in \mathcal{L} \), are twice differentiable at \( x^* \), and for each \( z^* \in C(x^*) \), there exist \( u^* \in U, v^* \in \mathbb{R}^r \), and \( w^* \in \mathbb{R}^r \) such that (2.1)- (2.4) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:

(a).

1. For each \( i \in \mathbb{L} \), \( \xi \rightarrow \prod_i (\xi, u^*_i, v^*_i, w^*_i, \lambda^*_i) \) is strictly \((F, \beta, \phi, \pi, \rho, \theta, m)\) - pseudosounivex at \( x^*_i, \phi \) is increasing, and \( \phi(0) = 0 \).
2. For each \( t \in M \setminus L, \xi \rightarrow \lambda_i(\xi, v^*_i, w^*_i, \lambda^*_i) \) is \((F, \beta, \phi, \pi, \rho, \theta, m)\) - sounivex at \( x^*_i, \phi \) is increasing, and \( \phi(0) = 0 \).
3. \( \sum_{i=1}^m \rho_i (x, x^*) \geq 0 \) for all \( x \in F \).

(b).

1. For each \( t \in L \), \( \xi \rightarrow \Pi_i (\xi, u^*_i, v^*_i, w^*_i, \lambda^*_i) \) is strictly \((F, \beta, \phi, \pi, \rho, \theta, m)\) - pseudosounivex at is increasing, and \( \phi(0) = 0 \).
2. For each \( t \in M \setminus L, \xi \rightarrow \lambda_i(\xi, v^*_i, w^*_i, \lambda^*_i) \) is strictly \((F, \beta, \phi, \pi, \rho, \theta, m)\) - pseudosounivex at \( x^*_i, \phi \) is increasing, and \( \phi(0) = 0 \).
3. \( \sum_{i=1}^m \rho_i (x, x^*) \geq 0 \) for all \( x \in F \).

(c).

1. For each \( t \in L \), \( \xi \rightarrow \Pi_i (\xi, u^*_i, v^*_i, w^*_i, \lambda^*_i) \) is strictly \((F, \beta, \phi, \pi, \rho, \theta, m)\) - pseudosounivex at \( x^*_i, \phi \) is increasing, and \( \phi(0) = 0 \).
2. For each \( t \in M \setminus L, \xi \rightarrow \lambda_i(\xi, v^*_i, w^*_i, \lambda^*_i) \) - pseudosounivex at \( x^*_i, \phi \) is increasing, and \( \phi(0) = 0 \).
3. \( \sum_{i=1}^m \rho_i (x, x^*) \geq 0 \) for all \( x \in F \).

(d).

1. For each \( t \in L \), \( \xi \rightarrow \Pi_i (\xi, u^*_i, v^*_i, w^*_i, \lambda^*_i) \) is strictly \((F, \beta, \phi, \pi, \rho, \theta, m)\) - pseudosounivex at \( x^*_i, \phi \) is increasing and \( \phi(0) = 0 \).
2. For each \( t \in M \setminus L, \xi \rightarrow \lambda_i(\xi, v^*_i, w^*_i, \lambda^*_i) \) - pseudosounivex at \( x^*_i, \phi \) is increasing, and \( \phi(0) = 0 \).
3. \( \sum_{i=1}^m \rho_i (x, x^*) \geq 0 \) for all \( x \in F \).
2. For each $t \in M \setminus L$, $\xi \rightarrow \Lambda_t(\xi, v^*, w^*, \lambda^*)$ is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-pseudosounivex at $x^*$, is increasing, and $\phi(0) = 0$;

3. $\sum_{t \in M} \rho_t(x, x^*) \geq 0$ for all $x \in F$;

e.

1. For each $t \in L \neq \phi$, $\xi \rightarrow \Pi_t(\xi, u^*, v^*, w^*, \lambda^*)$ is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-pseudosounivex at $x^*$, is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-pseudosounivex at $x^*$, is prestrictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-quasisounivex at $x^*$, and for each $t \in L$, is increasing and $\phi(0) = 0$, where $(L_1, L_2)$ is a partition of $L$;

2. For each $t \in M \setminus L$, $\xi \rightarrow \Lambda_t(\xi, v^*, w^*, \lambda^*)$ is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-quasisounivex at $x^*$, is increasing, and $\phi(0) = 0$;

3. $\sum_{t \in M} \rho_t(x, x^*) \geq 0$ for all $x \in F$;

f.

1. For each $f \in L$, $\xi \rightarrow \Pi_f(\xi, u^*, v^*, w^*, \lambda^*)$, is prestrictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-quasisounivex at $x^*$, is increasing, and $\phi(0) = 0$;

2. For each $t \in M \setminus L$, $\xi \rightarrow \Lambda_t(\xi, v^*, w^*, \lambda^*)$ is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-quasisounivex at $x^*$, is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-quasisounivex at $x^*$, is strictly $(F, \beta, \phi, \pi, \rho, \theta, m)$-quasisounivex at $x^*$, and for each $t \in L$, is increasing and $\phi(0) = 0$, where $(\{M_{L_1}, \{M_{L_2}\})$ is a partition of $M_{L}$;

3. $\sum_{t \in M} \rho_t(x, x^*) \geq 0$ for all $x \in F$;

4. $L_1 \neq 0$ or $M_{L_1} \neq 0$ or $\sum_{t \in M} \rho_t(x, x^*) > 0$.

Then $x^*$ is an optimal solution of (P).

Proof (a)

Suppose to the contrary that $x^*$ is not an optimal solution of (P). This implies that

$$f_i(\overline{x}) - \lambda_i^* g_i(\overline{x}) < 0,$$

for some $i \in P$. Since $u^* \geq 0$ and $u^* \neq 0$, we see that for each $t \in L$, $\sum_{i \in P} (f_i(\overline{x}) - \lambda_i^* g_i(\overline{x})) \leq 0$.

Now using this inequality, we see that

$$\Pi_{i \in P}(f_i(\overline{x}) - \lambda_i^* g_i(\overline{x})) = \sum_{i \in P} (f_i(\overline{x}) - \lambda_i^* g_i(\overline{x})) \leq 0$$

(by the feasibility of $\overline{x}$)

$$\leq 0 \ (by \ (2.12))$$

(by (2.3), (2.4), and the feasibility of $x^*$)

$$= \Pi_{i \in P}(f_i(\overline{x}) - \lambda_i^* g_i(\overline{x}))$$

and hence $\phi_i(\Pi_{i \in P}(x^*, u^*, v^*, w^*, \lambda^*)) = \Pi_{i \in P}(x^*, u^*, v^*, w^*, \lambda^*) \leq 0$.

Summing over and using the sublinearity of $F(x, x^*)$ we obtain

$$F(\pi(x, x^*), (\beta, \phi, \pi, \rho, \theta, m)) \leq \sum_{i \in P} (\phi_i(\Pi_{i \in P}(x^*, u^*, v^*, w^*, \lambda^*)) \leq 0,$$

which in view of (i) implies that

$$F(\pi(x, x^*), (\beta, \phi, \pi, \rho, \theta, m)) \leq \sum_{i \in P} (\phi_i(\Pi_{i \in P}(x^*, u^*, v^*, w^*, \lambda^*)) \leq 0,$$

As shown in the proof of Theorem 2.1, for each $t \in M / L$, $\Lambda_t(\pi, v^*, w^*), \Lambda_t(x^*, v^*, w^*) \leq t \in M / L, \Lambda_t(\pi, v^*, w^*), \Lambda_t(x^*, v^*, w^*) \leq 0,$

and so

$$\phi(\Lambda_t(\pi, v^*, w^*) = \Lambda_t(x^*, v^*, w^*) \leq 0,$$

which in view of (ii) implies that

$$\phi(\Lambda_t(\pi, v^*, w^*) \leq \Lambda_t(x^*, v^*, w^*) \leq 0,$$

and hence

$$\phi(\Lambda_t(\pi, v^*, w^*) \leq \Lambda_t(x^*, v^*, w^*) \leq 0,$$

which in view of (ii) implies that

$$\phi(\Lambda_t(\pi, v^*, w^*) = \Lambda_t(x^*, v^*, w^*) \leq 0,$$

As shown in the proof of Theorem 2.1, for each $t \in M / L$, $\Lambda_t(\pi, v^*, w^*), \Lambda_t(x^*, v^*, w^*) \leq 0,$

and so

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and so

$$\phi(\Lambda_t(\pi, v^*, w^*) = \Lambda_t(x^*, v^*, w^*) \leq 0,$$
\[-\frac{1}{2}\left(z^* + \sum_{i \in M} \sum_{k \in K} w_i^k \nabla H_k(x^*)z^* \right) = \sum_{i \in M} \rho_i(x^*, \lambda^*) \|\theta(x^*, \lambda^*)\|^{\alpha}.
\] (2.14)

Combining (2.13) and (2.14) and using (iii), we obtain

\[
F(x, x^*) = \sum_{i \in M} u_i^p \nabla G_i(x^*) + \sum_{k \in K} v^k \nabla H_k(x^*)z^* \geq \sum_{i \in M} u_i^p \nabla G_i(x^*) + \sum_{k \in K} v^k \nabla H_k(x^*)z^*.
\]

Now if we multiply by \(1/2\) and \(\beta\) to both sides of the resulting equation, using the sublinearity of \(F(x, x^*)\), and adding this equation to (2.2), we obtain the following inequality:

\[
\frac{1}{2} \nabla \beta(x, x^*) + \sum_{i \in M} \sum_{k \in K} w_i^k \nabla H_k(x^*)z^* \geq \sum_{i \in M} u_i^p \nabla G_i(x^*) + \sum_{k \in K} v^k \nabla H_k(x^*)z^*.
\]

Concluding Remarks

Based on Dinkelbach’s parametric model [1], we have established numerous sets of higher order parametric sufficient optimality criteria for a discrete mimax fractional programming problem using a variety of generalized \((F, \beta, \phi, \pi, \rho, \theta, m)\) - sonvexity constraints. These optimality models can be applied for constructing various duality models as well as for developing new algorithms for the numerical solution of mimax fractional programming problems. Furthermore, the obtained results generalize most of the results available in the current literature, and are application-oriented in the sense of interdisciplinary research. More importantly, duality models do have a significant role in the semiinfinite aspects of the mathematical programming, for example, the following semiinfinite mimax fractional programming problem:

\[
\text{Minimize } \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}
\]

subject to \(G_i(x, s) \leq 0\) for all \(s \in S_j, j \in \mathcal{J}, H_k(x, s) = 0\) for all \(s \in S_k, k \in \mathcal{K}, x \in X\), where \(X, \mathcal{F}, \mathcal{G}\) are as defined in the description of \((P)\), for each \(j \in \mathcal{J}\), \(\xi \rightarrow G_j(\xi, t)\) and \(\xi \rightarrow T_j\) and \(\xi \rightarrow S_k\) are compact subsets of complete metric spaces, for each \(\xi\), is a real-valued function defined on \(X\) for all \(t \in T_j, \xi \rightarrow H_k(\xi, s)\) for each \(j \in \mathcal{J}\) is a real-valued function defined on \(X\) for all \(s \in S_k, k \in \mathcal{K}, x \in X\) and \(s \rightarrow H_k(x, s)\) are continuous real-valued functions defined, respectively, on \(T_j\) and \(S_k\) for all \(x \in X\).

References