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A Novel Simplified Methodology for Solving the Stochastic Fractional Differential Equation

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Abstract

In this mini review, a novel yet simple methodology for solving the stochastic fractional differential equation has been proposed. Compared to the others, our methodology has been found to be much simpler.

Keywords: Fractional derivative; Stochastic fractional differential equation; Wiener process; Vector stochastic differential equation; Turbulence; Nonlinear type; Linear type; Drift term; Diffusion term; Riemann-Liouvielle type; Gamma function; Integer; Optimum approximation; Numerically; Recursive manner

Abbreviations: SFDE: Stochastic Fractional Differential Equation; SDE: Stochastic Differential Equation

Introduction

The stochastic fractional differential equation (SFDE) has been often cited in various disciplines e.g. turbulence, heterogeneous flows and materials etc. [1]. Unfortunately, solving the SFDE can be a rather complicated task. Therefore, a novel methodology for solving the SFDE has been proposed in this work. The proposed methodology is to firstly convert the SFDE to its equivalent vector stochastic differential equation (SDE) and solving the obtained equivalent SDE in a usual manner. Comparing to the previous ones [1-3], our methodology has been found to be much simpler. Moreover, it is also applicable to the SFDE of both linear and nonlinear type.

The proposed methodology

The SFDE with fractional derivative and non-fractional Wiener process [2], can be generally given by

$$\frac{d^{\alpha}}{dt^{\alpha}}X(t)=f\left(t,X(t)\right)+g\left(t,X(t)\right)\frac{d}{dt}W(t) \qquad (1)$$

Where α, $f\left(t,X(t)\right)$, $g\left(t,X(t)\right)$ and $W(t)$ stand for the order of the fractional derivative, the drift term and diffusion term of the SFDE and the Wiener process respectively. It should be mentioned here that $0<\alpha<1$. Moreover, $f\left(t,X(t)\right)$ and $g\left(t,X(t)\right)$ can be either linear or nonlinear functions. By assuming that the fractional derivative in (1) is of the Riemann-Liouvielle type [4], and applying the approximation of such fractional derivative [5], we have found that the fractional derivative of $X(t)$ can be given in term of the 1st order one as

$$\frac{d^{\alpha}}{dt^{\alpha}}X(t)\approx A(\alpha,N)t^{-\alpha}X(t)+B(\alpha,N)t^{1-\alpha}\frac{d}{dt}X(t)-\sum_{p=2}^{N}C(\alpha,p)t^{1-p-\alpha}Y_p(t) \qquad (2)$$

Where

$$A(\alpha,N)=\frac{1}{\Gamma(1-\alpha)}\left[1+\sum_{p-2}^{N}\frac{\Gamma\left(p-1+\alpha\right)}{\Gamma(\alpha)(p-1)!}\right]. \qquad (3)$$

$$B(\alpha,N)=\frac{1}{\Gamma(2-\alpha)}\left[1+\sum_{p-1}^{N}\frac{\Gamma\left(p-1+\alpha\right)}{\Gamma(\alpha)(p-1)!}\right].$$

$$C(\alpha,N)=\frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\frac{\Gamma\left(p-1+\alpha\right)}{(p-1)!}.$$

Noted that $\Gamma(\ )$ stands for the gamma function [6] and $N\geq 2$ where N must be strictly integer. Moreover, $Y_2(t),Y_3(t),\ldots,Y_N(t)$ which are the moments of $X(t)$, can be defined as follows

$$\frac{d}{dt}Y_2(t)=-X(t)$$

$$\frac{d}{dt}Y_3(t)=-2tX(t) \qquad (4)$$

$$\frac{d}{d}Y_N(t)=(1-N)t^{N-2}X(t)$$

Noted also that $Y_2(0)=0,Y_3(0)=0,\ldots,Y_N(0)=0$. By incorporating (2) to (1), we have

$$\frac{d}{dt}X(t)=-\frac{A\left(\alpha,N\right)t^{-\alpha}X(t)+\sum_{p=2}^{N}C\left(\alpha,p\right)t^{1-p-\alpha}Y_p(t)+f\left(t,X(t)\right)}{B\left(\alpha,N\right)t^{1-\alpha}}+\frac{g\left(t,X(t)\right)}{B\left(\alpha,N\right)t^{1-\alpha}}\frac{d}{dt}W(t) \qquad (5)$$

By combining (4) and (5), the vector SDE equivalent of (1) can be obtained as follows

$$d\mathbf{X}(t)=\left[\mathbf{A}(t)\mathbf{X}(t)+\mathbf{b}(t)\right]dt+\mathbf{C}(t)d\mathbf{W}(t) \qquad (6)$$

Where

$$\mathbf{X}(t)=\left[X(t)\quad Y_2(t)\quad Y_3(t)\quad \ldots\quad Y_N(t)\right]^T \qquad (7)$$

$$\mathbf{A}(t)=\begin{bmatrix} -\frac{A(\alpha,N)t^{-\alpha}}{B(\alpha,N)t^{1-\alpha}} & \frac{C(\alpha,2)t^{-1-\alpha}}{B(\alpha,N)t^{1-\alpha}} & \frac{C(\alpha,3)t^{-2-\alpha}}{B(\alpha,N)t^{1-\alpha}} & \cdots & \frac{C(\alpha,N)t^{1-N-\alpha}}{B(\alpha,N)t^{1-\alpha}} \\ -1 & 0 & 0 & 0 & 0 \\ -2t & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-N)t^{N-2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b}(t)=\begin{bmatrix} \frac{f(t,X(t))}{B(\alpha,N)t^{1-\alpha}} & 0 & 0 & \ldots & 0 \end{bmatrix}^T$$

$$\mathbf{C}(t)=\begin{bmatrix} \frac{g(t,X(t))}{B(\alpha,N)t^{1-\alpha}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$W(t)=\begin{bmatrix} W(t) & 0 & 0 & \ldots & 0 \end{bmatrix}^T$$

Therefore the following solution can be obtained

$$\mathbf{X}(\tau)=\mathbf{\Phi}(t,0)\mathbf{X}(0)+\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{b}(\tau)d\tau+\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{C}(\tau)d\mathbf{W}(\tau) \quad (8)$$

Where

$$\mathbf{\Phi}(t,0)=\mathbf{I}+\sum_{n=1}^{\infty}\left[\int_0^t \ldots \int_0^t \int_0^t \mathbf{A}(t_1)\mathbf{A}(t_2)\ldots\mathbf{A}(t_n)dt_n dt_{n-1}\ldots dt_1\right] \quad (9)$$

Noted that $0<t_n<t_{n-1}<\ldots<t_2<t_1<t$. Moreover, the stochastic integral terms in (8) i.e. $\int_0^t \mathbf{\Phi}(t,\tau)\mathbf{C}(\tau)d\mathbf{W}(\tau)$ is the Ito integral [7]. From $X(t)$, $X(t)$ which is the solution of (1), can be determined by the following equation

$$X(t)=X[1,1] \quad (10)$$

For determining $X(t)$ numerically, the Euler-Maruyama numerical approximation scheme [8] has been found to be of our interested due to its simplicity. Noted that the strong order of convergence i.e. $\gamma=0.5$ [8], must be chosen. Moreover, $N=7$ is recommended as this value practically provides the optimum approximation of the fractional derivative [5]. After applying such scheme, $X(t)$ can be numerically solved in a recursive manner as follows

$$\mathbf{X}(t_{n+1})=\mathbf{X}(t_n)+\left[\mathbf{A}(t_n)\mathbf{X}(t_n)+\mathbf{b}(t_n)\right](t_{n+1}-t_n)+\mathbf{C}(t_n)\left[\mathbf{W}(t_{n+1})-\mathbf{W}(t_n)\right] \quad (11)$$

Then $X(t)$ can be found by also using (10).

Conclusion

A novel simple methodology for solving the SFDE has been proposed. Such methodology is applicable to the SFDE of both linear and nonlinear type. Compared to the previous ones, our methodology has been found to be more simplified.

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