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AHybrid Solutions Method for Solving One Dimensional Parabolic Partial Differential Equations

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Abstract

A new continuous numerical method based on polynomials approximation is here proposed for solving the equation arising from heat transfer along a copper rod and a hollow tube subject to initial and boundary conditions. The method results from discretization of the heat equation which leads to the production of a system of algebraic equations. By solving the system of algebraic equations we obtain the problem approximate solutions.

Keywords: Polynomials; Interpolation; Multistep collocation; Parabolic partial differential Equations; Numerical method; One dimensional; Hybrid solutions; Algebraic equations; Heat equation; Heat conduction equation; Interpolation; Collocation; Temperature; Heat flow; Polynomials; Numerical accuracy; Evaporates; Zero; Ethyl alcohol; Initial temperature; Interpolation point.

Introduction

The development of continuous numerical techniques for solving heat conduction equation in science and engineering subject to initial and boundary conditions is a subject of considerable interest. In this paper, we develop a new numerical method which is based on interpolation and collocation at some point along the coordinates [1-3]. To do this we let $U(x,t)$ represents the temperature at any point in the rod and the tube. Heat is flowing from one end to another under the influence of temperature gradient $\partial U/\partial x$. To make a balance of the rate of heat flow in and out of the media, we consider R for thermal conductivity of the steel, C the heat capacity which we assume constants, and ρ the density and D the thermal diffusivity of alcohol. Heat flow in the rod is given by

$$-RA\frac{\partial U}{dx}-\left[-RA\left[\frac{\partial U}{dx}+\frac{\partial}{\partial x}\left(\frac{\partial U}{dx}\right)dx\right]\right]=C\rho\left(Adx\right)\frac{\partial U}{dt} \qquad (1.0)$$

Heat flow through the tube is given by

$$-DB\frac{\partial U}{dx}-\left[-DB\left[\frac{\partial U}{dx}+\frac{\partial}{\partial x}\left(\frac{\partial U}{dx}\right)dx\right]\right]=C\rho\left(Adx\right)\frac{\partial U}{dt} \qquad (1.1)$$

Where A and B are the cross sections of the rod and the tube respectively. Our new method strives to provide solutions to the heat flow eqns. (1.0) and (1.1).

The solution method

To set up the solution method we select an integer N such that $N>0$. We subdivide the interval $0\le x\le X$ into N equal subintervals with mesh points along the space axis given by $x_i = ih$, $i=\frac{1}{\beta}\left(\frac{1}{\beta}\right)N$ where $hN = X$. Similarly, we reverse the roles of x nd t, we select another integer M such that $M>0$. We also subdivide the interval $0\le t\le T$ into M equal subintervals with mesh points along time coordinate given by $t_j = jk$, $j=\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)M$, where $kM = T$, and h,k are the mesh sizes along space and time coordinates [4-6]. We seek for the approximate solution $\overline{U}(x,t)$ to $U(x,t)$ of the form

$$\overline{U}(x,t)\approx\overline{U}_{p-1}(x,t)=\sum_{r=0}^{p-1}a_r\left[q_r(x,t)+s_r(x,t)\right],\ x\in\left[x_i,x_{i+h}\right],\ t\in\left[t_j,t_{j+k}\right] \qquad (2.0)$$

over $h>0,\ k>0$ mesh sizes, such that $0=x_0<\ldots<x_i<\ldots<x_N$, $0=t_0<\ldots<t_j<\ldots<t_M$. We denote p to be the sum of interpolation points along the space and time coordinates respectively. That is $p=g+b$, where g is the number of interpolation points along the space axis, while b is the number of interpolation points along the time coordinate [7]. The bases functions q_r, s_r $r=0,1,\ldots,p-1$ are the Taylor's and Legendre's polynomials which are known, a_r are the constants to be determined. The interpolation values $\overline{U}_{i,j},\ldots,\overline{U}_{i+h-1,j}$ are assumed to have been determined from previous steps, while the method seeks to obtain $\overline{U}_{i+h,j}$

Applying the above interpolation conditions on eqn. (2.0) we obtain

$$\overline{U}_{i+h,j+k}(x,t)=a_0\left(q_0+s_0\right)\left(x_{i+h},t_{j+k}\right)+a_1\left(q_1+s_1\right)\left(x_{i+h},t_{j+k}\right)+\ldots+a_{p-1}\left(q_{p-1}+s_{p-1}\right)\left(x_{i+h},t_{j+k}\right) \qquad (2.1)$$

We let $h=-\frac{1}{\beta}\left(\frac{1}{\beta}\right)\left[g-\left(\frac{2\beta-1}{\beta}\right)\right]$ arbitrarily and $k=0$, then by Cramer's rule, eqn. (2.1) becomes

$$W\underline{a}=\underline{F},\qquad \underline{F}=\left(U_{v,j},U_{v+\frac{1}{\beta},j},\dots,U_{z,j}\right)^{T},\tag{2.2}$$

$$\underline{a}=\left(a_0,\dots,a_{p-1}\right)^{T}$$

and

$$W=\begin{bmatrix}(q_0+s_0)(x_v,t_j), & (q_1+s_1)(x_v,t_j), & ,\dots, & (q_{p-1}+s_{p-1})(x_v,t_j)\\ (q_0+s_0)\left(x_{v+\frac{1}{\beta}},t_j\right), & (q_1+s_1)\left(x_{v+\frac{1}{\beta}},t_j\right), & \dots, & (q_{p-1}+s_{p-1})\left(x_{v+\frac{1}{\beta}},t_j\right)\\ \dots, & \dots, & ,\dots, & \dots\\ (q_0+s_0)(x_z,t_j), & (q_1+s_1)(x_z,t_j), & ,\dots, & (q_{p-1}+s_{p-1})(x_z,t_j)\end{bmatrix}$$

Where

$z=i+g-\left(\frac{2\beta-1}{\beta}\right)$, $v=i-\frac{1}{\beta}$ and W^{-1} exist. Hence, from eqn. (2.2), we obtain

$$\underline{a}=\varpi\underline{F},\qquad \varpi=W^{-1}.\tag{2.3}$$

The vector $\underline{a}=\left(a_0,\dots,a_{p-1}\right)^{T}$ is now determined in terms of known parameters in $\varpi\underline{F}$. If ϖ_{r+1} is the $(r+1)^{th}$ row of ϖ, then

$$a_r=\underline{\varpi}_{r+1}\underline{F}\tag{2.4}$$

Eqn. (2.4) determines the values of a_r. Let us take the first and second derivatives of eqn. (2.0) with respect to x

$$\overline{U}'(x,t)=\sum_{r=0}^{p-1}\left[a_r\left(q_r'(x,t)+s_r'(x,t)\right)\right]$$

$$\overline{U}''(x,t)=\sum_{r=0}^{p-1}\left[a_r\left(q_r''(x,t)+s_r''(x,t)\right)\right]\tag{2.5}$$

Substituting eqn. (2.4) into eqn. (2.5) we obtain

$$\overline{U}''(x,t)=\sum_{r=0}^{p-1}\left[\underline{\varpi}_{r+1}\underline{F}\left(q_r''(x,t)+s_r''(x,t)\right)\right]\tag{2.6}$$

We reverse the roles of x and t in eqn. (2.1) and we arbitrarily set $k=0\left(\frac{1}{\alpha}\right)\left[b-\left(\frac{\alpha-1}{\alpha}\right)\right]$, $h=0$, again by Cramer's rule, eqn. (2.1) become

$$Y\underline{a}=\underline{E},\qquad \underline{E}=\left(U_{i,\eta-\frac{1}{\alpha}},U_{i,\eta},\dots,U_{i,\gamma}\right)^{T}\tag{2.7}$$

$$\underline{a}=\left(a_0,\dots,a_{p-1}\right)^{T}$$

and

$$Y=\begin{bmatrix}(q_0+s_0)\left(x_i,t_{\eta-\frac{1}{\alpha}}\right), & (q_1+s_1)\left(x_i,t_{\eta-\frac{1}{\alpha}}\right), & \dots, & (q_{p-1}+s_{p-1})\left(x_i,t_{\eta-\frac{1}{\alpha}}\right)\\ (q_0+s_0)(x_i,t_\eta), & (q_1+s_1)(x_i,t_\eta), & \dots, & (q_{p-1}+s_{p-1})(x_i,t_\eta)\\ \dots, & \dots, & \dots, & \dots\\ (q_0+s_0)(x_i,t_\gamma), & (q_1+s_1)(x_i,t_\gamma), & \dots, & (q_{p-1}+s_{p-1})(x_i,t_\gamma)\end{bmatrix}$$

where $\eta=j+\frac{1}{\alpha}$, $\gamma=j+b-\left(\frac{\alpha-1}{\alpha}\right)$ and Y^{-1} exist.

Hence, from eqn. (2.7) we obtain

$$\underline{a}=L\underline{E},\qquad L=Y^{-1}.\tag{2.8}$$

The vector $\underline{a}=\left(a_0,\dots,a_{p-1}\right)^{T}$ is now determined in terms of known parameters in $L\underline{E}$. If L_{r+1} is the $(r+1)^{th}$ row of L, then

$$a_r=\underline{L}_{r+1}\underline{E}\tag{2.9}$$

Also, eqn. (2.9) determines the values of a_r explicitly.

Taking the first derivatives of eqn. (2.0) with respect to t we obtain

$$\overline{U}'(x,t)=\sum_{r=0}^{p-1}\left[a_r\left(q_r'(x,t)+s_r'(x,t)\right)\right]\tag{2.10}$$

Substituting eqn. (2.9) into eqn. (2.10) we obtain

$$\overline{U}'(x,t)=\sum_{r=0}^{p-1}\left[\underline{L}_{r+1}\underline{E}\left(q_r'(x,t)+s_r'(x,t)\right)\right]\tag{2.11}$$

But by eqns. (1.0) and (1.1) it is obvious that eqn. (2.11) is equal to eqn. (2.6), therefore,

$$\sum_{r=0}^{p-1}\left[\underline{L}_{r+1}\underline{E}\left(q_r'(x,t)+s_r'(x,t)\right)\right]-\sum_{r=0}^{p-1}\left[\underline{\varpi}_{r+1}\underline{F}\left(q_r''(x,t)+s_r''(x,t)\right)\right]=0\tag{2.12}$$

Collocating eqn. (2.12) at $x=x_i$ and $t=t_j$ produces a new numerical scheme that solves equations (1.0) and (1.1) explicitly.

Numerical examples

In this section, we will test the numerical accuracy of the new method by using the new scheme to solve two examples. That is, we compute an approximate solutions of eqns. (1.0) and (1.1) at each time level. To achieve this, we truncate the polynomials after second degree and the average is used as the basis function in the computation [8-10]. The resultant scheme is used to solve the following problems.

Example 1

A hollow tube 20 cm long is initially filled with air containing 2% of ethyl alcohol vapors [2]. At the bottom of the tube is a pool of alcohol which evaporates into the stagnant gas above. (Heat transfers to the alcohol from the surroundings to maintain a constant temperature of 30 °C, at which temperature the vapour pressure is 0.1 atm.) At the upper end of the tube, the alcohol vapors dissipate to the outside air, so the concentration is essentially zero. Considering only the effects of molecular diffusion, determine the concentration of alcohol as a function of time and the distance x measured from the top of the tube.

Molecular diffusion follows the law

$$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2},$$

Where, D is the diffusion coefficient, with units of cm²/sec. (This is the same as for the ratio k/p, which is often termed thermal diffusivity.) For ethyl alcohol, D = 0.119 cm²/sec at 30 °C , and the vapor pressure is such that 10 volume percent alcohol in air is present at the surface [11-15]. The initial condition is $c(x,0)=2.0$, and boundary conditions are $c(0,t)=0,\ c(20,t)=10$. Subdivide the length of the tube into five intervals, so that Δx = 4cm. Using the maximum value permitted for Δt yields

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$$D\frac{\Delta t}{(\Delta x)^2}=1,\quad 0.119\frac{\Delta t}{4^2}=1,\ \Delta t=134.45$$

Taking $\beta = 4$, $\alpha = 2$ implies that $v = i-\frac{1}{4},\ z = i+\frac{1}{4}\ and\ \eta=\gamma=j+\frac{1}{2}$. For $i=\frac{1}{4},\frac{1}{2},\frac{3}{4},\ldots,\ and\ j=\frac{1}{2},\frac{1}{6},\frac{1}{2},\ldots,$ and by taking two interpolation points along space coordinate and one interpolation point along time coordinate implies that $g=2,\ b=1,\ \Rightarrow\ p=3$, and this simply means that $h=-\frac{1}{4},0,\frac{1}{4}\ \ and\ k=0,\frac{1}{2}$, then the calculated concentration of alcohol is as shown in Table 1 [15-20].

Table 1: Concentrations of alcohol.

T	$x=0$	$x=4$	$x=8$	$x=12$	$x=16$	$x=20$
0	0	2	2	2	2	10
134.45	0	1.75	2	2	6	10
268.9	0	1.56	1.97	3.5	7	10
403.35	0	1.42	2.11	3.75	6.94	10
537.8	0	1.33	2.23	3.94	6.92	10
672.25	0	1.28	1.67	4.1	6.93	10

Example 2

Solve for the temperature in a copper rod 1.2 cm long, with the outer curved surface insulated so that heat flows in on one direction [21-24]. If the initial temperature $({}^{o}C)$ within the rod are given by

$$U=100x\ \ for\ 0\le x\le 0.6,\qquad U=100(1.2-x),\ \ for\ \ 0.6\le x\le 1.2$$

Find the temperature as a function of x and t if both faces are maintained at 0° C. For copper, $k=0.37$, $\rho = 0.433$. We use $\Delta x = 0.2\ cm$, we then find Δt by the relation

$$\frac{k\Delta t}{\rho\,(\Delta x)^2}=\frac{1}{4},\ \ \Delta t=0.01$$

Taking $\beta = 4$, $\alpha = 2$ implies that $v=i-\frac{1}{4},\ z=i+\frac{1}{4}\ and\ \eta=\gamma=j+\frac{1}{8}$. For $i=\frac{1}{4},\frac{1}{2},\frac{3}{4},\ldots,\ and\ j=\frac{1}{8},\frac{1}{4},\frac{3}{8},\ldots,$ and by taking two interpolation points along space coordinate and one interpolation point along time coordinate implies that $g=2,\ b=1,$ and $p=3$, this simply gives $h=-\frac{1}{4},0,\frac{1}{4}\ \ and\ k=0,\frac{1}{8}$, then the calculated temperatures are as shown in Table 2 [25-27].

Table 2: Calculated temperatures.

t	$x=0$	$x=0.2$	$x=0.4$	$x=0.6$	$x=0.8$	$x=1.00$	$x=1.20$
0	0	20	40	60	40	20	0
0.01	0	20	40	55	40	20	0
0.02	0	20	39.38	51.25	39.38	20	0
0.03	0	19.92	38.44	48.28	38.44	19.92	0
0.04	0	19.75	37.36	45.82	37.36	19.75	0
0.05	0	19.48	36.22	43.71s	36.22	19.48	0

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